M33R - Complex Analysis

Problem Paper 1 - due 29 January 2007

1. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors must be zero.
   Hint: Try to prove a similar result for two factors $z_1 z_2 = 0$, and then write $(z_1 z_2) z_3 = 0$.

2. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

   and then solving a pair of simultaneous equations in x and y.
   Hint: Since no real number satisfies the equation $z^2 + z + 1 = 0$, we can assume that $y \neq 0$.

3. Prove the following identities:

   (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

   (b) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$,

   (c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

4. Show that the equation $|z - z_0| = R$ of a circle, centred at z_0 with radius R, can be written

$$|z|^2 - 2\Re(z\overline{z_0}) + |z_0|^2 = R^2.$$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

   (a) $5i/(2 + i) = 1 + 2i$,

   (b) $(-1 + i)^7 = -8(1 + i)$.

6. Show that if $\Re(z_1) > 0$ and $\Re(z_2) > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$

7. Show that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$