

PUERTO RICAN MATHEMATICAL OLYMPIAD

ROUND I

2007/2008

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Preface

Each year, many countries around the world participate in an International Mathematical Olympiad. This is a mathematics problem-solving competition in which high school students tackle significant problems over a two-day period. Many countries also participate in one of several Regional Olympiads held each year. For example, approximately 17 countries are expected to participate in the Central American and Caribbean Regional Mathematical Olympiad to be held next year. This will take place in Mayaguez, Puerto Rico, from May 26-June 1, 2010. Teams of high school students from throughout the region will compete for individual honors and participate in several cultural events during this Olympiad. The University of the West Indies, Mona Campus, has accepted an invitation to sponsor a Jamaican Mathematical Olympiad Team to participate in this event. During the 2009/2010 school year, the Department of Mathematics will be organizing activities which allow us to select national champions at each grade level and form a national team to participate in this exciting event.

In order to assist us in preparing for this opportunity, the Alliance for the Improvement of Mathematics Learning (AFAMaC) has kindly provided us with many materials. This alliance is based in Puerto Rico and represents a joint effort between the Department of Mathematical Sciences, University of Puerto Rico, Mayaguez Campus, and the Puerto Rico Department of Education. The materials contained in these pages consist of the questions and answers used in the first two stages of the Puerto Rican Mathematical Olympiad held during the 2007/2008 school year.

Each stage of the Olympiad was held at Levels 1 and 2. Level 1 was open to students in Grades 4–6 and Level 2 was open to students in Grades 7–12. This year, here in Jamaica, we will only be inviting high school students to participate in the Mathematical Olympiad. However, we are including in this manual the Puerto Rican questions for Grades 4–6 because they still have a great value. For students and teachers who have no experience with Olympiad questions, they provide a good introduction to the style and content of the questions likely to be used in future competitions. And although students will be sure to get many of them right, many of them will surely generate much discussion and real thought.

In the process of translating the questions into English, a few changes were made in order to take into account the new context in which they would be used. For example, almost all of the original questions were multiple choice; in this manual only a very few are. In some cases, the British units of measure (*e.g.* feet) were changed to metric units (*e.g.*, metres). Also, although the questions are the same (except for any accidental changes that may have occurred), the solutions are sometimes presented in a different style. In all cases, the changes were made to make the material seem as natural as possible in the Jamaican context. Any errors in the translation, the resulting English grammar, or the mathematical content of the material are entirely my own responsibility. Students and teachers who have questions on the solutions given or have found mistakes in this manual are invited to let me know of these matters.

On behalf of the Department of Mathematics, University of the West Indies, Mona Campus, I thank the AFAMaC for assisting us in many ways as we prepare for this year's events. In particular, I thank them for their kind permission to use their past questions and circulate them freely in Jamaica. I also wish all students good luck and much success in their mathematical studies this year.

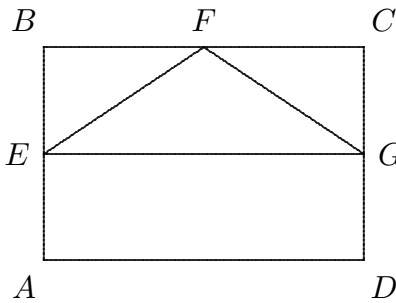
Raymond McEachin

September 21, 2009

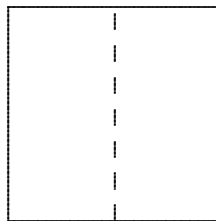
- 10) In the sum below, different letters stand for different digits, AH is a two-digit number, and HEE is a three-digit number. Find $H + E$.

$$\begin{array}{r} AH \\ + A \\ \hline HEE \end{array}$$

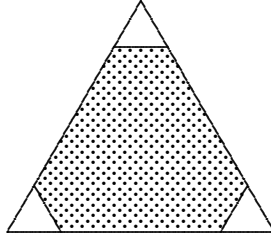
- 11) What is the least number of colours needed to paint a cube in such a way that no two adjacent sides have the same colour?
- 12) In the diagram below, the rectangle $ABCD$ has area 36 square units. The points E , F , and G are the midpoints of the sides on which they lie. What is the area of the triangle EFG ?



- 13) If a cube 5 cm on each side is formed of cubes 1 cm on each side, how many of the smaller cubes cannot be seen from the outside?
- 14) Luis wrote down all of the numbers from 1 to 100. What is the total number of digits Luis wrote down?
- 15) If the base of a triangle is increased by 10% and its altitude is decreased by 10%, by how much is its area changed? [You may give your answer as a percent.]
- 16) One light bulb flashes every 2 minutes and another flashes every $3\frac{1}{2}$ minutes. If both flash at midnight, when is the first time after 1 am that they flash together again?
- 17) A square sheet of paper is folded down the middle to form two rectangles as shown below. If each rectangle has perimeter 18 cm, what is the perimeter of the original square?



- 18) At the Balboa Theatre, the price of admission is \$7. They have four shows each day. For the first and second shows, admission is half price. One Thursday, the fourth show had twice as many patrons as the third, and the third had twice as many as the first and second shows combined. If they received \$1183 that day, how many patrons were at the fourth show?
- 19) From an equilateral triangle with perimeter 75 cm, an equilateral triangle with sides 5 cm is removed from each corner. What is the perimeter of the resulting region?



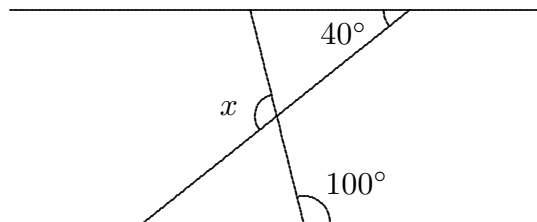
- 20) Susana works in a building with 5 offices. In that building, there are three plants: a cactus, an azalea, and an orchid. Each day, Susana changes the plants in the offices. In how many ways can she arrange the plants so that not all three of them are placed in the same office?

First Round Questions
Level II (Grades 7–12)

- 1) On a typical piano, there are 87 keys and $\frac{1}{3}$ of them are black while the rest are white. How many are white?
- 2) Conchita worked hard and doubled her savings. With this money, she paid off \$600 she owed and had \$200 left. How much savings did she start with?
- 3) Antonio is the uncle of Rosa; Ana is the niece of Pedro; Fabiola is the sister of Antonio; Antonio is the father of Pedro. How are Rosa and Pedro related?
- 4) Consider 5 whole numbers. How many of them are odd if their product is odd?
- 5) In the figure below, the letters A, B, C, \dots, G represent numbers (not necessarily distinct). If the sum of any three consecutive numbers is 18, what is the value of F ?

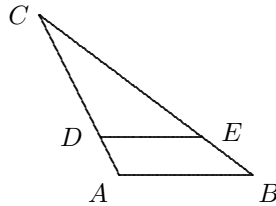
3	A	B	C	D	8	E	F	G
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- 6) How many positive divisors does $5 \times 4 \times 3 \times 2$ have? (The divisors of a number include 1 and the number itself.)
- 7) In the diagram below, the horizontal lines are parallel. What is the measure of $\angle x$?



- 8) What is the sum of all the digits in the number $10^{2007} - 2007$?
- 9) Maria gets channels 2 through 42 on her television. When scrolling through the channels they cycle so that after channel 42 comes channel 2 again. If she is watching Channel 15 and advances through 518 channels, which one will she be watching?
- 10) Pablo bought an item on sale. After a 15% discount its price was \$106.25. What was its original price?
- 11) Simon wrote 3 on the chalkboard. Then he erased it and wrote its square, 9, in its place. Then he erased it and wrote its square, 81, in its place. If he repeats this process 2007 times, what will be the last digit of the final number?

- 12) How many ways are there of ordering the letters $L, A, P, I,$ and Z so that the first and last letters are vowels?
- 13) There are some marbles in a bag. Maria said, “There are only 3 marbles in the bag and all of them are black”. Luis said, “There are only two black marbles and two red marbles in the bag”. Jorge said, “There are only black marbles in the bag”. If only one of them is wrong, how many marbles are in the bag?
- 14) In the diagram below, DE is parallel to AB . If the area of $\triangle DEC$ is $\frac{3}{4}$ the area of $\triangle ABC$ and AC is 1 m, what is the length of DC ?

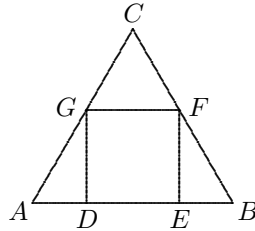


- 15) Some vandals stole all the tires off the cars and motorcycles in a certain street. The police arrested them and determined that 44 vehicles (cars and motorcycles) were vandalized. If a total of 144 tires were stolen, how many motorcycles were in the street?
- 16) The diagonals of a rhombus are in the proportion $3 : 4$ and their sum is 56 cm. What is the perimeter of the rhombus?
- 17) The rows and columns of an 8×8 square board are numbered from 1 to 8 as shown below. In each square, Maurice placed as many checkers as the sum of the row and column it was in. How many checkers did Maurice place all together?

	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6								
7								
8								

- 18) Consider all the four-digit numbers which can be formed using the digits 3, 4, 6, and 7 without repetition. How many of these numbers are divisible by 44?

- 19) In the triangle below, ABC is an equilateral triangle and $DEFG$ is a square. If AB is 1 m, what is the length of DE ?

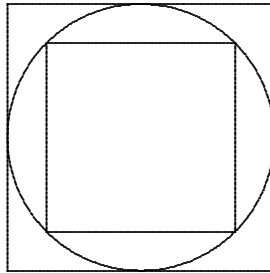


- 20) If x is the solution to the equation

$$\frac{x+1}{1} + \frac{x+2}{2} + \dots + \frac{x+100}{100} = 100$$

then what is x ?

- 21) In the figure below a small square is inscribed inside a circle and a larger square is circumscribed about the circle. What is the ratio between the areas of the larger and smaller squares?



- 22) On a certain island, there are only two types of people: those who always tell the truth and those who always lie. Three inhabitants of the island were talking. Andrea said, “Barbara always tells the truth”. Barbara said, “Andrea and Carlos always tell the truth”, and Carlos said, “Andrea lies”. It follows that:
- a) The three of them tell the truth;
 - b) Andrea and Barbara tell the truth and Carlos lies;
 - c) Andrea tells the truth and Barbara and Carlos lie;
 - d) Andrea and Barbara lie and Carlos tells the truth;
 - e) The three of them lie.
- 23) Suppose 50 distinct numbers are chosen from $\{1, 2, 3, \dots, 100\}$. If their sum is 3000, what is the least number of even numbers that are possible among the 50?
- 24) Consider a certain set of 20 consecutive numbers greater than 50. What is the largest number of primes possible in the set?

- 25) A student passed a certain number of examinations with an average of 23. After he sat one more exam, his average was 22.25. What are the possible scores on his last exam if his score on every exam was between 18 and 30 (including 18 and 30)?
- 26) An equilateral triangle has the same perimeter as a rectangle with dimensions b and h (with $b > h$). If the area of the triangle is $\sqrt{3}$ times that of the rectangle, what is the value of b/h ?
- 27) Suppose Q is a cube and S is a sphere whose center is at a vertex of Q and radius is equal to a side length of Q . What is the volume of the intersection of Q and S as a proportion of the volume of Q ?
- 28) Let $P(x) = x^3 + ax^2 + bx + c$ and suppose the sum of two of the roots of the polynomial is 0. Which of the following relationships is certain?
- a) $a \times b \times c = 0$
 - b) $c = a \times b$
 - c) $c = a + b$
 - d) $b^2 = a \times c$
 - e) None of the above
- 29) Let ABC be an isosceles triangle with $AB = BC \neq AC$, and let P be a point on AB . In how many ways can a point Q be chosen so that $\triangle APQ$ is similar to $\triangle ABC$?
- 30) Alberto, Barbara, Clara, and David are playing a game with 40 cards from a standard deck of cards. After one deal Alberto exclaimed, "How unusual, I have no spades in my hand!" If 10 of the 40 cards were spades, what was the probability that Barbara doesn't have any spades in her hand either? (You may give your answer in terms of factorials and other expressions without simplifying.)