**First Round Solutions**

**Level I (Grades 4–6)**

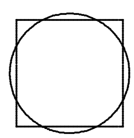
1) There are four basic rectangles not made up of smaller ones. There are three more rectangles made up of two smaller ones each, two rectangles made up of three smaller ones each, and the outer rectangle (which is in fact a square) made up of the four smaller ones. Thus there are $4+3+2+1=10$ rectangles in all.

2) The first, third, fifth, etc., terms are 1, 2, 3, etc. The second, fourth, sixth, etc., terms are 4, 8, 12, etc. Following these progressions separately, the next two terms are 20 and 6.

3) Reading from right to left, the there are two choices for the last digit, two for the middle digit, and one for the first digit (which must be a 5). Then there are $1\times 2\times 2=4$ such numbers in total, which are 500, 505, 550, and 555.

4) If half a pound is removed from each box then a total of 10 lbs is removed from the 20 boxes. The remaining papayas will weigh 1590 lbs.

5) In the diagram below, the square intersects the circle in 8 points. This is the maximum because no side of a square can cut a circle in more than two points. Since squares have four sides, no square can intersect a circle in more than eight points.



6) If there are ten trees then there are nine spaces between them. Since each space is 15 m, the distance between the first and last trees is 135 m.

7) Suppose Maria gives a gift to Ana. Then Ana cannot give one to Maria or else Rosa must give a gift to herself. So Ana must give a gift to Rosa and Rosa must give one to Maria. If Maria doesn't give a gift to Ana then she must give one to Rosa. In this case, Rosa will give one to Ana and Ana will give one to Maria. Thus there are two ways of exchanging gifts at the party.

8) If there are $n$ dogs and $n$ chickens then there are $4n$ legs on the dogs and $2n$ legs on the chickens. The total number of legs is $6n$. The only possible number of legs is 24 because that is the only 6-multiple among the choices offered.

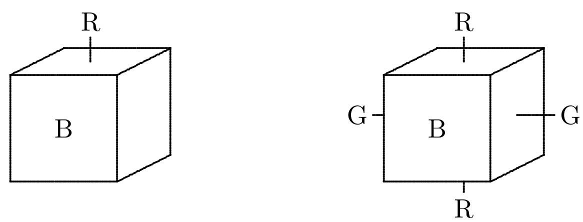
9) After $n$ years, Mario was $27+n$ and Pedro was $3+n$. Mario would have been three times as old as Pedro when $27+n=3(3+n)$. In this case,

$$27+n=9+3n; \qquad 18=2n; \qquad 9=n$$

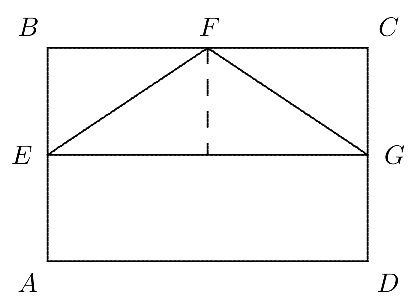
Thus after 9 years Mario was three times as old as Pedro, and this happened when Mario was 36 and Pedro was 12.

10) Note that $AH$ can be no larger than 99 and $A$ can be no larger than 9. Then $HEE$ must be a number between 100 and 108. In particular, $H = 1$. Also, $A = 9$ because if it were less than 9 then the sum of $AH$ and $A$ would be less than or equal to 89. Then $A = 9$, $AH = 91$, and $HEE = 100$. Thus $H + E = 1$.

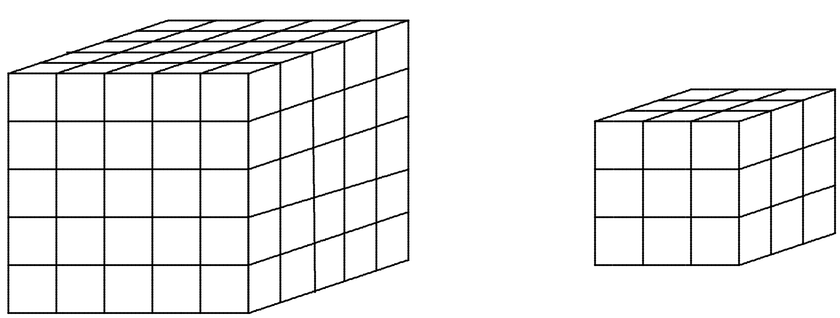
11) Suppose one side, say the top, is red and an adjacent side, say the front, is blue. A side adjacent to both of them must be another colour such as green. However, with these three colours the rest of the cube can be coloured as shown. (The back of the cube would be blue.) The least number of colours needed is 3.




12) The area of rectangle $BCGE$ is half the area of $ABCD$ and the area of triangle $EFG$ is half the area of $BCGE$. Then the area of the triangle $EFG$ is 9 square units.




13) If you remove all the small cubes which can be seen from the outside, there would remain a $3 \times 3 \times 3$ cube which could not have been seen from the outside. Thus there are 27 smaller cubes which cannot be seen from the outside.

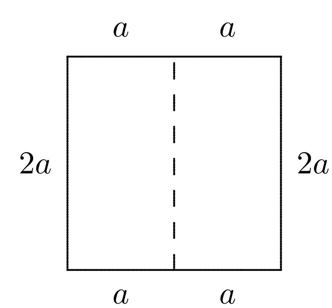


14) Luis wrote down 1 digit for each number from 1 to 9, 2 digits for each number from 10 to 99, and three digits for the number 100. Thus he wrote down $9(1)+90(2)+1(3) = 192$ digits in total.

15) Let $b$ and $h$ be the original base and height, respectively, and $B$ and $H$ the new ones. Then $B = b + 0.1b = 1.1b$ and $H = h - 0.1h = 0.9h$. The new area is $\frac{1}{2}BH = \frac{1}{2}(1.1b)(0.9h) = \frac{0.99}{2}bh$. Since the original area is $\frac{1}{2}bh$, the new area is 0.99 or 99% of the original one.

16) The first bulb will flash after 2, 4, 6, 8, 10, 12, 14, …, minutes. The first time on this list that is a multiple of $3\frac{1}{2}$ minutes is 14 minutes. It follows that the bulbs flash together every 14 minutes. This means that they flash together at 12:14 am, 12:28 am, 12:42 am, 12:56 am, and 1:10 am. The first time after 1 am that they flash together is 1:10 am.

17) Referring to the diagram below, $a+2a+a+2a = 18$. Thus $6a = 18$ and hence $a = 3$. It follows that the perimeter of the original square is 24

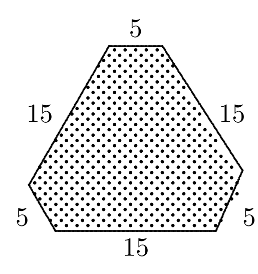



18) Let $A$, $B$, $C$, and $D$ be the number of patrons at the four shows that day. Then $C = 2(A+B)$ and $D = 2C = 4(A+B)$. The money the theatre received was

$$3.50A + 3.50B + 7C + 7D = 3.50(A+B) + 14(A+B) + 28(A+B) = 45.50(A+B)$$

Thus $45.5(A+B) = 1183$ and hence $A+B = 26$. There were $4(26) = 104$ patrons at the fourth show.

19) Each side of the original triangle is 25 cm long. After removing the smaller triangles each longer side has length 15 cm. Since each shorter side has length 5 cm, the perimeter of the resulting region is 60 cm.



20) There are two possibilities to consider: three offices contain one plant each or one office contains two plants while another has one. In the first case, one plant may be placed in any of 5 offices, another placed in any of the 4 remaining offices, and the third in any of the last three offices. Thus there are $5 \times 4 \times 3 = 60$ ways of placing the plants this way. In the second case, there are three ways of pairing two plants to be placed in the same office: the cactus and the azalea, the cactus and the orchid, and the azalea and the orchid. Then the pair may be placed in any of 5 offices and the single plant placed in any of the remaining 4 offices. Then there are $3 \times 5 \times 4 = 60$ ways of placing the plants in this case. Combining the two cases, there are $60 + 60 = 120$ ways all together of placing the plants.

## First Round Solutions

## Level II (Grades 7–12)

1) There are $\frac{2}{3} \times 87 = 58$ white keys on a piano.

2) Since Conchita had \$200 left after paying off \$600 she owed, she had \$800 on hand. Since this is twice her original savings, she started off with \$400.

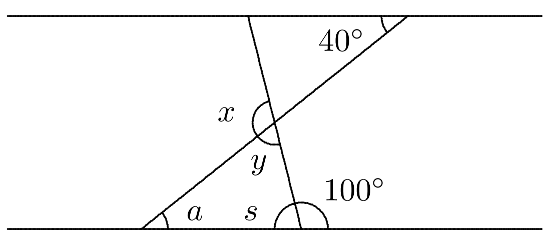
3) Since Antonio is the uncle of Rosa and the father of Pedro, Rosa and Pedro are cousins.

4) If a single one of the 5 numbers is even then their product is even. Since their product is odd, all 5 numbers are odd.

5) Since $3+A+B = 18$ and $A+B+C = 18$, it follows that $C = 3$. Since $C+D+8 = 18$ it follows that $D = 7$. Since $D + 8 + E = 18$, we have $E = 3$. Since $8 + E + F = 18$ we have $F = 7$.

6) First, $5 \times 4 \times 3 \times 2 = 120$. Its divisors are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120. It has 16 divisors. For an alternate solution, note that the prime factorization of 120 is $2^3 \times 3 \times 5$. A factor of 120 may contain 0, 1, 2, or 3 twos, 0 or 1 three, and 0 or 1 five. Then there are $4 \times 2 \times 2 = 16$ factors of 120.

7) Since $a$ is opposite an angle of $40°$, it is $40°$. Since $s$ is the supplement for a $100°$ angle, it is $80°$. Since $a + s + y = 180°$, $y = 60°$. Since $x$ and $y$ are supplements, $x = 120°$.




8) To subtract 2007 from $10^{2007}$ it is easiest to subtract 1 first and then subtract 2006 more. Since the numeral for $10^{2007}$ is 1 followed by 2007 zeros, subtracting 1 gives the number with 2007 digits which are all 9. Subtracting 2006 from this number gives the number with 2007 digits with 9 repeated 2003 times followed by 7993. Then the sum of the digits is $9(2003) + 7 + 9 + 9 + 3 = 18,055$.

9) Each cycle through the loop consists of 41 channels. So she will be back to Channel 15 after advancing through 41, 82, 123, 164, …, 492 channels. This leaves 26 channels to go, and Maria will end up watching Channel 41.

10) If $P$ is the original price then $P - 0.15P = 106.25$. Thus $0.85P = 106.25$ and so $P = \$125$.

11) If we note only the last digits of the numbers Simon wrote, we obtain 3, 9, 1, 1, 1, 1, .... Then after 2009 times, the last digit is 1.

12) There are 2 choices for the first letter: A or I. There are 3 choices for the second one: L, P, or Z. Then there are 2 choices for the third letter, 1 choice for the fourth one, and 1 choice for the last one. Thus there are $2 \times 3 \times 2 \times 1 \times 1 = 12$ possible orderings.

13) If Maria is wrong then both Luis and Jorge would be right. Since they contradict each other, this is impossible. Then Maria is right, Jorge is right, and Luis is wrong. Since Maria is right, there are 3 marbles in the bag.

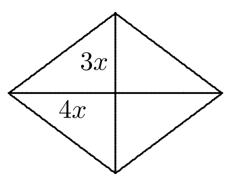
14) Since the triangles shown are similar,

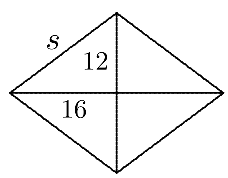
$$\left(\frac{DC}{AC}\right)^2 = \frac{3}{4}; \qquad \left(\frac{DC}{1}\right)^2 = \frac{3}{4}; \qquad (DC)^2 = \frac{3}{4}.$$

Then $DC = \frac{\sqrt{3}}{2}$ m.

15) If $c$ is the number of cars and $m$ the number of motorcycles in the street then $c+m = 44$ and $4c + 2m = 144$. By eliminating one of the variables, one determines that $c = 28$ and $m = 16$. In particular, there were 16 motorcycles in the street.

16) if $3x$ and $4x$ are the distances (in centimeters) as shown, then the sum of the diagonals is $6x + 8x = 14x = 56$. Then $x = 4$. If $s$ is the length of a side as shown, then $s^2 = 12^2 + 16^2 = 400$ and hence $s = 20\,\text{cm}$. Then the perimeter of the rhombus is 80 cm.

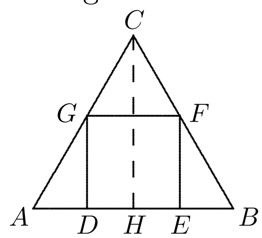



17) In the first row, Maurice placed $2 + 3 + 4 + \cdots + 9 = 44$ checkers. In the second one, he placed $3 + 4 + 5 + \cdots + 10 = 52$ checkers. In the third row, he placed $4 + 5 + 6 + \cdots + 11 = 60$ checkers. Continuing in this way, he placed 68, 76, 84, 92, and 100 checkers in rows 4, 5, 6, 7, and 8, respectively. Then the total number of checkers he placed is $44 + 52 + 60 + \cdots + 100 = 576$

18) In order to have a number divisible by 44 it must be divisible by both 4 and 11. Also, a number is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. Thus the only numbers we can form which are divisible by 4 are 3764, 7364, 4736, 7436, 3476, and 4376. Among these numbers, the only ones divisible by 11 are 7436 and 3476. These two numbers are the only ones divisible by 44.

19) Let $H$ be the point shown in the diagram below. Since $AB$ is 1 m, $AH = 1/2$ m and




$HC = \sqrt{3}/2$ m. Also, if $s$ denotes the side length of the square then $AD = (1-s)/2$ and $DG = s$. Since the triangles $ADG$ and $AHC$ are similar,

$$\frac{\frac{1}{2} - \frac{s}{2}}{s} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}; \qquad \frac{1-s}{2s} = \frac{1}{\sqrt{3}}$$

By cross multiplying, we obtain

$$\sqrt{3} - \sqrt{3}s = 2s; \qquad \sqrt{3} = (2+\sqrt{3})s; \qquad s = \frac{\sqrt{3}}{2+\sqrt{3}}$$

20) We have

$$\left(\frac{x}{1} + \frac{1}{1}\right) + \left(\frac{x}{2} + \frac{2}{2}\right) + \left(\frac{x}{3} + \frac{3}{3}\right) + \cdots + \left(\frac{x}{100} + \frac{100}{100}\right) = 100$$

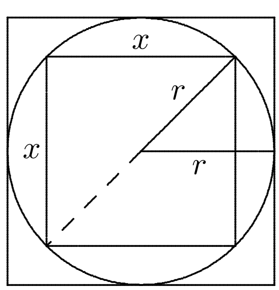
$$\left(\frac{x}{1} + 1\right) + \left(\frac{x}{2} + 1\right) + \left(\frac{x}{3} + 1\right) + \cdots + \left(\frac{x}{100} + 1\right) = 100$$

After canceling 100 from both sides,

$$x + \frac{x}{2} + \frac{x}{3} + \cdots + \frac{x}{100} = 0; \qquad x\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}\right) = 0$$

It follows that $x = 0$.

21) Let $r$ be the radius of the circle. The larger square has side length $2r$ and hence has area $(2r)^2 = 4r^2$. If $x$ is the side length of the smaller square then, by the Pythagorean theorem, $x^2 + x^2 = (2r)^2$. Thus $2x^2 = 4r^2$ and hence $x^2 = 2r^2$. Then the area of the square is $x^2 = 2r^2$. The ratio of the larger area to the smaller one is $\frac{4r^2}{2r^2} = 2$.



22) Suppose first that Andrea tells the truth. Then according to her statement, Barbara always tells the truth and, according to Barbara's statement, Carlos tells the truth. Finally, according to Carlos's statement, Andrea lies. Since this is a contradiction, Andrea does not tell the truth. This means that Barbara lies but Carlos tells the truth. The correct answer is that Andrea and Barbara lie and Carlos tells the truth.

23) Suppose we try to use no even numbers at all. Then we would have to choose 1, 3, 5, …, 99, but their sum is only 2500. If we try to use one even and 49 odd numbers the resulting sum will be odd. Thus we must use at least two even numbers. If we try to use two even numbers and 48 odd numbers the largest possible sum would be obtained by replacing 1 and 3 with 98 and 100. In this case the new sum will be $2500 - 4 + 198 = 2694$. So we need to replace at least two more odd numbers with evens. Doing this, we will obtain the largest possible sum by replacing 5 and 7 with 94 and 96. In this case, the new sum will be $2694 - 12 + 190 = 2872$. So we still need at least two more even numbers. We can remove 9 and 11 and replace them with any even numbers whose sum is 148. For example, we could use 92 and 56. Thus the smallest number of even numbers required is 6, and this can be achieved by using $\{56, 92, 94, 96, 98, 100\} \cup \{13, 15, 17, \ldots, 99\}$.

24) Among the 20 consecutive numbers 97, 98, 99, …, 116, we find the following six primes: 97, 101, 103, 107, 109, and 113. We show that this is the maximum number of primes that can be found. Note first that any 20 consecutive numbers must contain 10 even numbers. These are multiples of two and hence composite. Among the 10 remaining odd numbers, at least three of them must be multiples of 3. These must also be composite. Also, among the 10 odd numbers, there must be two multiples of 5; one of them, but not both, can be a multiple of 3 which has already been counted. Then there must be:

    10 numbers which are multiples of 2;

    3 numbers which are multiples of 3 but not multiples of two;

    1 number which is a multiple of 5 but not a multiple of 2 or 3.

This means that any 20 consecutive numbers must contain at least 14 distinct numbers which are composite. Therefore, at most 6 can be prime.

25) Suppose his average after $n$ exams was 23 and after $n+1$ exams was 22.25. If $t_n$ was his total after $n$ exams then $\frac{t_n}{n} = 23$ and hence $t_n = 23n$. If $\ell$ denotes his score on the last exam, then

$$\frac{t_n + \ell}{n+1} = 22.25; \qquad t_n + \ell = 22.25n + 22.25$$

Since $t_n = 23n$ we obtain $23n + \ell = 22.25n + 22.25$ and hence $\ell = -0.75n + 22.25$. Multiplying throughout by 4 we obtain $4\ell = -3n + 89$. Thus to summarize the situation, we know that $\ell$ is an integer, $18 \leq \ell \leq 30$, and $4\ell = -3n + 89$ for some positive integer $n$. By trial and error we see that $n = 3$ is the only possible choice for $n$, and in this case $\ell = 20$. Then the student scored 20 on his last exam.

26) Since the equilateral triangle has perimeter $2b+2h$, each of its sides has length $\frac{2}{3}(b+h)$. Also, its area is one-half the product of its base and height. Since its height is $\frac{\sqrt{3}}{2}$ of a side length, its area is

$$\frac{1}{2}\left(\frac{2}{3}(b+h)\right)\frac{\sqrt{3}}{2}\left(\frac{2}{3}(b+h)\right) = \frac{\sqrt{3}}{9}(b+h)^2$$

Since this area is $\sqrt{3}$ times that of the rectangle,

$$\frac{\sqrt{3}}{9}(b+h)^2 = \sqrt{3}bh; \qquad \frac{1}{9}(b+h)^2 = bh$$

Then $(b+h)^2 = 9bh$ and it follows that $b^2 - 7bh + h^2 = 0$. Dividing throughout by $h^2$,

$$\left(\frac{b}{h}\right)^2 - 7\left(\frac{b}{h}\right) + 1 = 0$$

Using the quadratic formula, one obtains $\frac{b}{h} = \frac{7 \pm 3\sqrt{5}}{2}$. The choice $\frac{b}{h} = \frac{7-3\sqrt{5}}{2}$ implies that $\frac{b}{h} < 1$ and hence $b < h$. Since this is impossible, $\frac{b}{h} = \frac{7+3\sqrt{5}}{2}$.

27) Let the radius of the sphere and the side length of the cube (both of which are equal) be denoted by $a$. Since the center of the sphere is at a vertex of the cube, exactly $\frac{1}{8}$ of the volume of the sphere is inside the cube. Then the volume of the intersection is

$$\frac{1}{8}\left(\frac{4}{3}\pi a^3\right) = \frac{\pi}{6}a^3$$

Since the volume of the cube is $a^3$, the proportion of its volume which is in the intersection is $\frac{\pi}{6}$.

28) Let $r_1$, $r_2$ and $r_3$ be the roots of the polynomial, with $r_1 + r_2 = 0$. Then, on one hand,

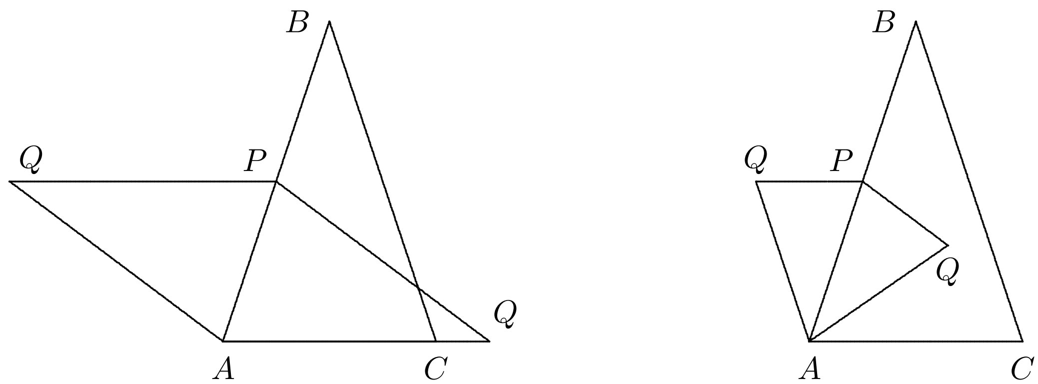
$$\begin{aligned} P(x) &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 \end{aligned}$$

Since $r_1 + r_2 = 0$, the coefficient for $x^2$ is $-r_3$. Also, the coefficient for $x$ is

$$r_1r_2 + (r_1 + r_2)r_3 = r_1r_2.$$

Thus $P(x) = x^3 - r_3x^2 + r_1r_2x - r_1r_2r_3$. On the other hand, since $P(x) = x^3 + ax^2 + bx + c$ we have $a = -r_3$, $b = r_1r_2$, and $c = -r_1r_2r_3$. Among the stated relations, the one that is certain is $a \times b = c$.

29) There are 4 locations for the point $Q$. Two are such that the angle $Q$ is the same as $B$, and two such that the angle $Q$ is the same as $A$ or $C$. The four locations are shown below.




30) Of the 30 cards outside of Albert's hand, 10 are spades and 20 are not. The number of hands without spades that Barbara could hold is

$$\binom{20}{10} = \frac{20!}{10!10!}$$

The total number of hands Barbara could hold is

$$\binom{30}{10} = \frac{30!}{10!20!}$$

The probability that Barbara holds no spades in her hand is

$$\frac{\left(\frac{20!}{10!10!}\right)}{\left(\frac{30!}{10!20!}\right)} = \frac{20!20!}{10!30!}$$