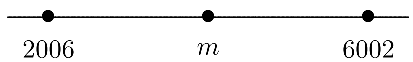
THE UNIVERSITY OF THE WEST INDIES, MONA

Presents

The 2012 Jamaican Mathematical Olympiad

Solutions for Grades 7 and 8

1. If m is half way between 2006 and 6002 (as shown in the figure below) then $m - 2006 = 6002 - m$. Thus $2m = 8008$ and so $m = 4004$.



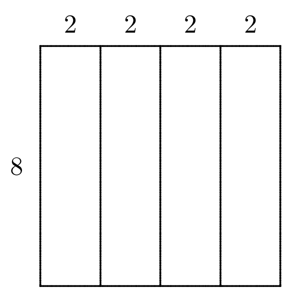
---

2. We have

$$\frac{2012+2012}{2012+2012+2012} = \frac{2012(1+1)}{2012(1+1+1)} = \frac{2012(2)}{2012(3)} = \frac{2}{3}$$

---

3. Since the area of the square is $64\,\text{cm}^2$, each side has length 8 cm. Then each smaller rectangle has width 2 cm and height 8 cm. Thus each smaller rectangle has perimeter (in centimetres) $2+8+2+8=20$.



---

4. The product of all the prime numbers less than 2012 has the form $2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times 2011$. (One may verify that 2011 is, indeed, prime.) Rearranging terms, this is

$$(2 \times 5) \times (3 \times 7 \times 11 \times \cdots \times 2011) = 10 \times (3 \times 7 \times 11 \times \cdots \times 2011)$$

The last digit of this product is clearly 0.

---

5. First, consider the times of the form $ab\!:cd$, where a, b, c, and d are the numbers 2, 0, 1, and 2 in some order. There are 6 ways of replacing two of the letters with 2, as follows:

$$22\!:cd, \qquad 2b\!:2d, \qquad 2b\!:c2, \qquad a2\!:2d, \qquad a2\!:c2, \qquad ab\!:22.$$

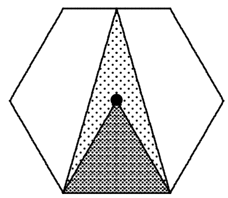
For each of these ways, we can choose the remaining letters to be 0 and 1 in two different ways. This gives $6 \times 2 = 12$ possibilities in all. They are:

$$\begin{array}{cccccc} 22\!:01, & 20\!:21, & 20\!:12, & 02\!:21, & 02\!:12, & 01\!:22, \\ 22\!:10, & 21\!:20, & 21\!:02, & 12\!:20, & 12\!:02, & 10\!:22. \end{array}$$

Since each of these displays occurs on an electronic watch exactly once between 00:00 and 23:59, the answer is 12.

---

6. Consider a triangle with the same base as the given triangle and with one vertex being the centre of the circle. (For example, consider the shorter, darker triangle in the figure below.) The area of the shorter triangle is 1/6 the area of the hexagon. The taller triangle



has twice as much area because it has the same base as the shorter one and twice its height. Then the proportion of the area of the hexagon which is contained in the taller triangle is

$$2\left(\frac{1}{6}\right) = \frac{2}{6} = \frac{1}{3}$$

---

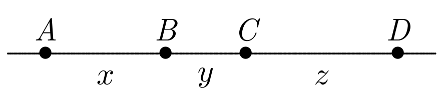
7. The total amount of sugar (in litres) in the mixture is $2 \times 0.10 + 3 \times 0.15 = 0.20 + 0.45 = 0.65$. The mixture itself has 5 litres of juice all together. The percentage of sugar in the juice is

$$\frac{0.65}{5} \times 100 = (0.13) \times 100 = 13.$$

The mixture contains 13% sugar.

---

8. Let $AB = x$, $BC = y$, and $CD = z$. From the information given, $x + y = 10$ and $x + y + z = 22$. It follows that $z = 12$. We also have $y + z = 15$ and it follows that $y = 3$. Thus $BC = 3$.



---

9. Suppose January 1 was on a Sunday. Then there would be five Sundays in January (the 1st, 8th, 15th, 22nd, and 29th), five Mondays, (the 2nd, 9th, 16th, 23rd, and 30th), and five Tuesdays (the 3rd, 10th, 17th, 24th, and 31st) in that month. Since there were only four

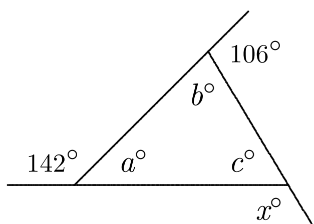
Mondays, January 1st was not on Sunday. Similarly, January 1 was not on Monday or there would have been five Mondays again. Also, January 1 was not on Wednesday, Thursday, or Friday or there would have been five Fridays in that January. Finally, it was not on Saturday or else there would have been five Mondays again. So the only possibility is that January 1 was on Tuesday. This works because when January 1 is on Tuesday there are five Tuesdays, five Wednesdays and five Thursdays in January. On the other hand, there are four Fridays, four Saturdays, four Sundays, and four Mondays in that month.

---

10. From the first two equations, $b = \frac{2}{a}$ and $b = \frac{3}{c}$. Then $\frac{2}{a} = \frac{3}{c}$ and it follows that $c = \frac{3a}{2}$. From the next two equations, $d = \frac{4}{c}$ and $d = \frac{5}{e}$. Then $\frac{4}{c} = \frac{5}{e}$ and it follows that $c = \frac{4e}{5}$. Since $c = c$ we have $\frac{4e}{5} = \frac{3a}{2}$. Then $8e = 15a$ and so $\frac{e}{a} = \frac{15}{8}$.

---

11. Let a, b, and c be the measures of the interior angles, as shown in the figure below. Since the three angles were formed by extending the sides of the triangle, $142 + a = 180$,



$106 + b = 180$, and $c + x = 180$. From the first two equations, $a = 38$ and $b = 74$. Furthermore, we know that $a + b + c = 180$. This means that $38 + 74 + c = 180$ and so $c = 68$. Since $x + c = 180$, it follows that $x = 112$.

---

12. Suppose one married couple is denoted by A_1 and A_2, another by B_1 and B_2, and the third by C_1 and C_2. The only way to form a three-person group with no married couple present is to choose one of the A's, one of the B's, and one of the C's for the group. There are two ways to choose one of the A's, two ways for the B's, and two for the C's. Thus there are $2 \times 2 \times 2 = 8$ ways of choosing in all. That is, there are eight ways of forming a three-person group in which there will not be a married couple.

---

13. To begin, one can spend all of the \$1500 to buy 15 boxes of chocolates. This gives a total of 15 coupons which may be redeemed for 5 more boxes. This gives another 5 coupons which may be redeemed for one box of chocolate with 2 coupons left over. Finally, the one box contains a coupon which may be redeemed with the other two for one last box of chocolate. The greatest number of boxes one can get for \$1500 is $15 + 5 + 1 + 1 = 22$.

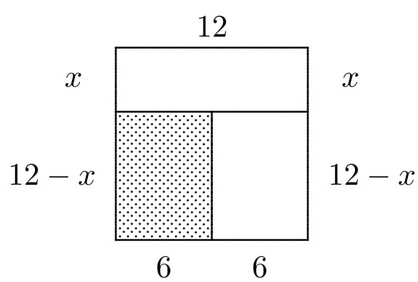
14. Let h be the hypotenuse of the triangle and a the length of the other side. By the Pythagorean theorem, $a^2 + \left(\sqrt{60}\right)^2 = h^2$ and hence $a^2 + 60 = h^2$. By the laws of algebra, $h^2 - a^2 = 60$ and so $(h+a)(h-a) = 60$. Since h and a are positive integers with $h > a$, $h+a$ and $h-a$ are also positive integers and $h+a > h-a$. Since $h+a$ and $h-a$ are factors of 60, the possibilities are:

$$\begin{array}{lll} h+a=60,\ \ h-a=1; & h+a=20,\ \ h-a=3; & h+a=12,\ \ h-a=5; \\ h+a=30,\ \ h-a=2; & h+a=15,\ \ h-a=4; & h+a=10,\ \ h-a=6. \end{array}$$

When $h+a = 30$ and $h-a = 2$ then $h = 16$ and $a = 14$. This gives a right triangle with sides 14 and $\sqrt{60}$ and hypotenuse 16. When $h+a = 10$ and $h-a = 6$ then $h = 8$ and $a = 2$. This gives a right triangle with sides 2 and $\sqrt{60}$ and hypotenuse 8. In the other pairs of equations, the solutions for h and a are not integers. Thus there are two right triangles meeting our criteria: 14, $\sqrt{60}$, and 16, and 2, $\sqrt{60}$ and 8.

---

15. The upper rectangle has width 12 and its height may be denoted by x, as shown in the figure below. (All dimensions are in centimetres.) The lower rectangles obviously have



equal height and it is given that they have equal perimeter. This implies that they have equal width as well, which is 6 in both cases. Since the upper and lower rectangles have the same perimeter,

$$\begin{aligned} 12 + x + 12 + x &= (12 - x) + 6 + (12 - x) + 6 \\ 24 + 2x &= 36 - 2x \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

This means that each lower rectangle has height 9. It follows that the area of the shaded rectangle is $9 \times 6 = 54$.

---

16. Using the usual multiplication algorithm,

$$111,111,111 \times 111,111,111 = 12,345,678,987,654,321$$

The sum of the digits in this product is $1 + 2 + 3 + \cdots + 8 + 9 + 8 + 7 + \cdots + 1 = 81$.

17. Let c be then number of chairs and s the number of stools in the room. For each chair there are 6 legs: four for the chair and two for the child sitting on it. For each stool there are 5 legs: three for the stool and two for the child sitting on it. The total number of legs is $6c + 5s$, and this is equal to 39. The possibilities for c are 1, 2, 3, 4, 5, and 6. (If there were 7 or more chairs there would be 42 or more legs, which is too many.) The only possibility that gives an integer number of stools is $c = 4$. In this case $s = 3$ and there are $6(4) + 3(5) = 24 + 15 = 39$ legs in the room.

---

18. There are 5 triangles congruent to the one highlighted in Figure 18(a), five congruent to the one in Figure 18(b), and so on. There are 25 isosceles triangles in all.

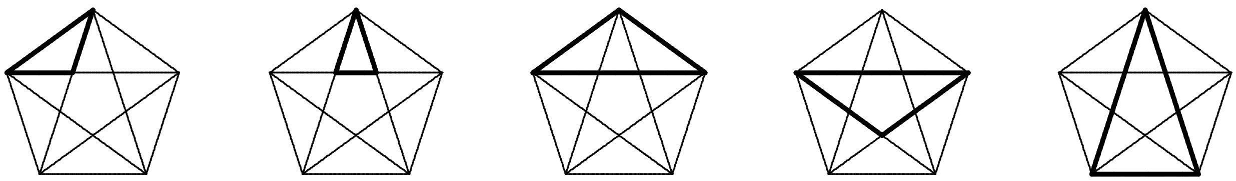


Fig. 18(a) Fig. 18(b) Fig. 18(c) Fig. 18(d) Fig. 18(e)

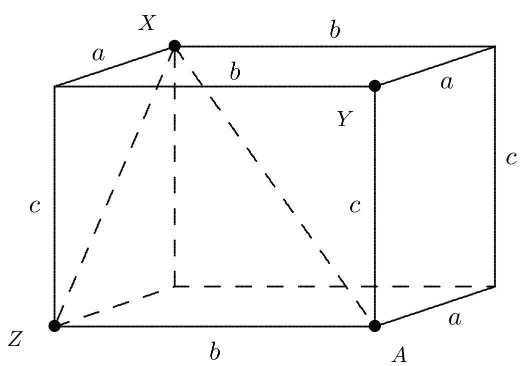
---

19. Let M and S be the current ages of Marco and Shawana. Since Marco was three times older than Shawana eighteen years ago, $M - 18 = 3(S - 18)$. Since Marco is now twice as old as Shawana, $M = 2S$. Then, by substitution,

$$2S - 18 = 3(S - 18); \qquad 2S - 18 = 3S - 54; \qquad 36 = S.$$

---

20. Let a, b, and c be the dimensions of the cuboid as shown. Since XZ is perpendicular to ZA, we have $(XZ)^2 + (ZA)^2 = (XA)^2$ by the Pythagorean theorem. Furthermore,



$(XZ)^2 = a^2 + c^2$ and $(ZA)^2 = b^2$. Thus $(XA)^2 = a^2 + b^2 + c^2$. Now note that $a^2 + b^2 = (XY)^2 = 64$, $a^2 + c^2 = (XZ)^2 = 55$, and $b^2 + c^2 = (YZ)^2 = 81$. Then, adding all three equations, $2a^2 + 2b^2 + 2c^2 = 64 + 55 + 81 = 200$. Thus $a^2 + b^2 + c^2 = 100$ and it follows that $XA = \sqrt{100} = 10$.

21. Since $b = 6$ the number in question has the form $a6cd$. Since the number is a multiple both of 4 and 5, it must be a multiple of 20. Thus it must have the form $a600$, $a620$, $a640$, $a660$, or $a680$. We consider each possibility in turn. Since a is the double of c, the number in the first case would be 0600. However, this is not a four-digit number and must be discarded. The number in the second case would be 4620. This is also a 3-multiple and so it satisfies all stated criteria. The number in the third case would be 8640. This is also 3-multiple and it also satisfies all stated criteria. The number in the last two cases is impossible because no single digit a can be the double of 6 or 8. Then there are two numbers that satisfy all stated criteria: 4620 and 8640.

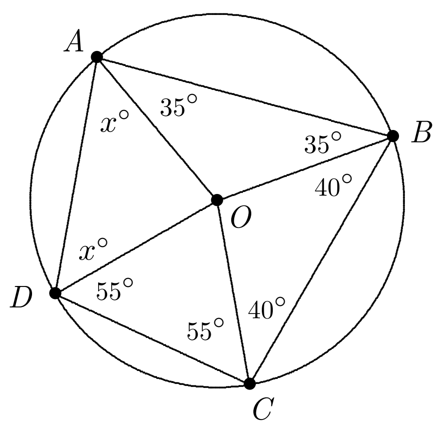
---

22. We may cross-multiply the given expression as follows:

$$\frac{2x - y}{x + y} = \frac{2}{3}; \qquad 3(2x - y) = 2(x + y); \qquad 6x - 3y = 2x + 2y; \qquad 4x = 5y; \qquad \frac{x}{y} = \frac{5}{4}.$$

---

23. Let $\angle ODA = x°$. Note that $OA = OB = OC = OD$ because OA, OB, OC, and OD are all radial segments. Then the triangles OAB, OBC, OCD, and ODA are all isosceles.



Thus $\angle OAB = \angle OBA = 35°$, $\angle OBC = \angle OCB = 40°$, $\angle OCD = \angle ODC = 55°$, and $\angle ODA = \angle OAD = x°$. Since the sum of the angles in any quadrilateral is 360°,

$$\begin{aligned} 2(35) + 2(40) + 2(55) + 2x &= 360 \\ 260 + 2x &= 360 \\ 2x &= 100 \\ x &= 50 \end{aligned}$$

That is, $\angle ODA = 50°$.

24. We first determine the number of positive integers less than 1000 which are in fact multiples of 5 or 7 (or both). The 5-multiples less than 1000 are 5, 10, 15, 20, ..., 995, and there are 199 such numbers in all. The 7-multiples less than 1000 are 7, 14, 21, 28, ..., 994, and there are 142 such numbers in all. However, some numbers are multiples both of 5 and 7, and these are the multiples of 35. The 35-multiples less than 1000 are 35, 70, 105, 140, ..., 980, and there are 28 such numbers in all. Then the number of positive integers less than 1000 which are multiples of 5 or 7 (or both) is $199 + 142 - 28 = 313$. Then the number which are not multiples of 5 or 7 is $999 - 313 = 696$.

---

25. Let

$$S_1 = \{1, 4, 7, 10, 13, 16, 19, 22, 25\},$$
$$S_2 = \{2, 5, 8, 11, 14, 17, 20, 23\},$$
$$S_3 = \{3, 6, 9, 12, 15, 18, 21, 24\}.$$

Then S_1 consists of all the numbers in S with remainder 1 when divided by 3, S_2 consists of all the numbers with remainder 2, and S_3 consists of all the numbers divisible by 3. Note that the sum of any two numbers in S_1 is *not* divisible by 3. Similarly, the sum of any two numbers in S_2 is *not* divisible by 3. However, the sum of any number in S_1 and any number in S_2 *is* divisible by 3. Suppose T is a set with the greatest number of elements that can be chosen from S so that the sum of any two elements is *not* divisible by 3. Then T can contain all the elements in S_1 or all the elements in S_2, but it cannot contain elements from both subsets. Furthermore, suppose T contains all the elements in S_1. Then it can also contain one element from S_3. Similarly, if T contains all the elements in S_2 it may also contain one element from S_3. However, T cannot contain two elements from S_3. For example if T contains 6 and 15 then it contains $6 + 15 = 21$, and this is a multiple of 3. Then the possibilities are that:

a) T contains all elements in S_1 and one element from S_3, or

b) T contains all elements in S_2 and one element from S_3.

Since (a) gives a set with more elements, T contains 10 elements. For example, we could have $T = \{1, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$