THE UNIVERSITY OF THE WEST INDIES, MONA

Presents

The 2012 Jamaican Mathematical Olympiad

Solutions for Grades 9, 10, and 11

1. There are 3 branches in the rosebush, $3 \times 3 = 9$ roses in the rosebush, and $2 \times 3 \times 3 = 18$ bees in the rosebush.

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2. Since 4 students study both languages, they should not be counted twice. Then the number who study at least one language is $17 + 13 - 4 = 26$.

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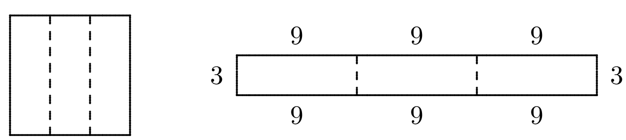
3. Since the square has area $81\,\text{cm}^2$, each side has length $9\,\text{cm}$. When the square is cut into three equal rectangles, each one is $3\,\text{cm} \times 9\,\text{cm}$. When the strips are placed end-to-end, the resulting rectangle is $27\,\text{cm} \times 3\,\text{cm}$. Its perimeter is $27\,\text{cm} + 3\,\text{cm} + 27\,\text{cm} + 3\,\text{cm} = 60\,\text{cm}$.



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4. Since one-fourth of Marsha's books are novels, the number of her books is a multiple of 4. Since one-ninth of them are cookbooks, the number is also a multiple of 9. Then the number of her books is also a multiple of 36. The 36-multiples are 36, 72, 108, ... The only one between 50 and 100 is 72. Then Marsha has 72 books.

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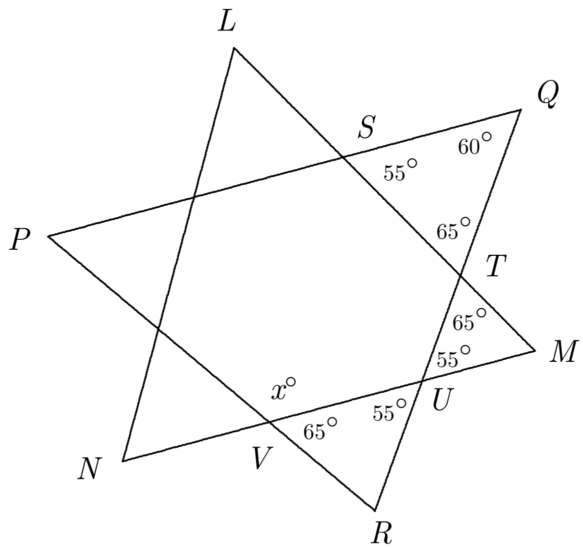
5. From the information given, $4^{xy} = (4^x)^y = 9^y = 256$. On the other hand, we also have $4^4 = 4 \times 4 \times 4 \times 4 = 256$. Then $4^{xy} = 4^4$ and it follows that $xy = 4$.

6. Let T, U, and V be the points of intersection as shown in the figure below. From the information given, the angles at Q, M, and R are each $60°$. Since the sum of the angles in



any triangle is $180°$, $\angle QTS = 65°$. By vertical angles, $\angle MTU = 65°$ and it follows that $\angle MUT = 55°$. By vertical angles again, $\angle RUV = 55°$ and it follows again that $\angle RVU = 65°$. Finally, since $\angle PVM$ and $\angle RVM$ are supplementary, $x + 65 = 180$. Then $x = 115$.

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7. Let n be the first of the four numbers. Then the other ones are $n+2$, $n+4$, and $n+6$. From the information given,

$$\begin{aligned} n + (n+2) + (n+4) + (n+6) &= 48 \\ 4n + 12 &= 48 \\ 4n &= 36 \\ n &= 9 \end{aligned}$$

The consecutive odd numbers are 9, 11, 13, and 15. The largest one is 15.

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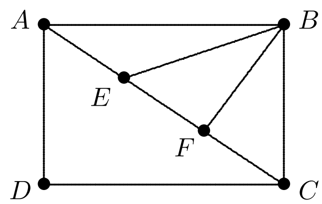
8. The two-digit perfect squares are 16, 25, 36, 49, 64, and 81. So suppose every pair of consecutive digits of a number is a perfect square. If the digit 1 is present the next digit must be 6, when 2 is present the next digit must be 5, and so on. Note that whenever 5, 7, or 9 occurs there cannot be a next digit. Among other things, this means that the first digit of the number cannot be 5, 7, or 9. We consider each of the possibilities 1, 2, 3, 4, 6, and 8 in turn. If the first digit of the number is 1 it has to be 1649. If its first digit is 2 it has to be 25. If its first digit is 3 it has to be 3649. If its first digit is 4 it has to be 49. If its first digit is 6 it has to be 649. And if its first digit is 8 it has to be 81649. The longest such number is 81649 and it has five digits.

9. From the information given, the rectangle $ABCD$ has area 15. It follows that the triangle ABC has area $15/2$. Furthermore, the triangles BAE, BEF, and BFC all have



equal bases and equal heights. Thus they have equal areas. Then the area of BEF is $1/3$ the area of ABC, and this is

$$\frac{1}{3} \times \frac{15}{2} = \frac{15}{6} = \frac{5}{2}.$$

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10. The easiest approach is to note that $1000 = 10^3$. Then $1000^{2012} = (10^3)^{2012} = 10^{6036}$. When written as a numeral, this number has a 1 followed by 6036 zeroes. It has 6037 digits in all.

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11. Suppose first that a, b, c, and d are any positive numbers. If $\frac{a}{b} > \frac{c}{d}$ then, multiplying both sides by the positive number bd, we have $ad > bc$. On the other hand, if $ad > bc$ then, multiplying both sides by $\frac{1}{bd}$, we have $\frac{a}{b} > \frac{c}{d}$. Thus we may say that

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad ad > bc$$

We now compare the five quantities (a)–(e). First, $\frac{a}{b-1} > \frac{a}{b+1}$ because $b - 1 < b + 1$ and when a is divided by a smaller quantity the quotient is larger. Second, $\frac{a}{b-1} > \frac{2a}{2b+1}$ if and only if

$$a(2b+1) > (b-1)(2a); \qquad 2ab + a > 2ab - 2a; \qquad a > -2a.$$

Since a is positive, this is true. Third, $\frac{a}{b-1} > \frac{2a}{2b-1}$ if and only if

$$a(2b-1) > (b-1)(2a); \qquad 2ab - a > 2ab - 2a; \qquad -a > -2a; \qquad a < 2a.$$

Since a is positive, this is true. Finally, $\frac{a}{b-1} > \frac{3a}{3b+1}$ if and only if

$$a(3b+1) > (b-1)(3a); \qquad 3ab + a > 3ab - 3a; \qquad a > -3a.$$

Since a is positive, this is true. In summary, the largest of the five quantities is $\dfrac{a}{b-1}$.

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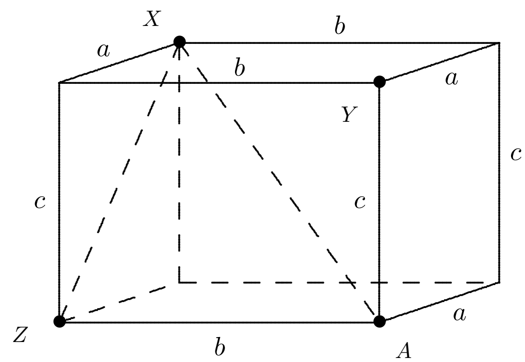
12. Let a, b, and c be the dimensions of the cuboid as shown. Since XZ is perpendicular to ZA, we have $(XZ)^2 + (ZA)^2 = (XA)^2$ by the Pythagorean theorem. Furthermore,



$(XZ)^2 = a^2 + c^2$ and $(ZA)^2 = b^2$. Thus $(XA)^2 = a^2 + b^2 + c^2$. Now note that $a^2 + b^2 = (XY)^2 = 64$, $a^2 + c^2 = (XZ)^2 = 55$, and $b^2 + c^2 = (YZ)^2 = 81$. Then, adding all three equations, $2a^2 + 2b^2 + 2c^2 = 64 + 55 + 81 = 200$. Thus $a^2 + b^2 + c^2 = 100$ and it follows that $XA = \sqrt{100} = 10$.

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13. In order to have a remainder of 5, the one-digit number dividing 1001 must be 6, 7, 8, or 9. If 1001 is divided by 6 the remainder is 5; if 1001 is divided by 7 the remainder is 0 (in other words, 7 divides evenly into 1001); if 1001 is divided by 8 the remainder is 1; and if 1001 is divided by 9 the remainder is 2. Therefore, the number dividing 1001 is 6. When 2012 is divided by 6, the remainder is 2.

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14. First, $\dfrac{ab-b^2}{ab-a^2} = \dfrac{b(a-b)}{a(b-a)} = \dfrac{-b(b-a)}{a(b-a)} = -\dfrac{b}{a}$. Then

$$\frac{a^2-b^2}{ab} - \frac{ab-b^2}{ab-a^2} = \frac{a^2-b^2}{ab} - \left(-\frac{b}{a}\right) = \frac{a^2-b^2}{ab} + \frac{b}{a} = \frac{a^2-b^2}{ab} + \frac{b^2}{ab} = \frac{a^2}{ab} = \frac{a}{b}.$$

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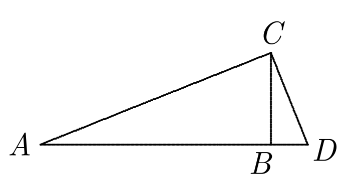
15. The triangles ABC and ACD are similar. This is because both are right triangles



with a common angle at A. Since the sum of the angles in any triangle is 180°,

$$90^\circ + \angle A + \angle ACB = 90^\circ + \angle A + \angle ACD.$$

It follows that $\angle ACB = \angle ACD$. Thus the two triangles have all of their corresponding angles equal. Since corresponding sides in similar triangles are in the same proportion,

$$\frac{AB}{AC} = \frac{AC}{AD}; \qquad \frac{64}{AC} = \frac{AC}{100}; \qquad (AC)^2 = 6400; \qquad AC = 80$$

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16. When the digits of a 3-digit number abc are put in reverse order the new number, cba, is larger if and only if $c > a$. The possibilities for a and c are:

$1b2$,	$1b3$,	$1b4$,	$1b5$,	$1b6$,	$1b7$,	$1b8$,	$1b9$,
	$2b3$,	$2b4$,	$2b5$,	$2b6$,	$2b7$,	$2b8$,	$2b9$,
		$3b4$,	$3b5$,	$3b6$,	$3b7$,	$3b8$,	$3b9$,
			$4b5$,	$4b6$,	$4b7$,	$4b8$,	$4b9$,
				$5b6$,	$5b7$,	$5b8$,	$5b9$,
					$6b7$,	$6b8$,	$6b9$,
						$7b8$,	$7b9$,
							$8b9$,

There are 36 possibilities listed, and for each one there are 10 choices for b. Then there are 360 such numbers in all.

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17. Let h be the hypotenuse of the triangle and a the length of the other side. By the Pythagorean theorem, $a^2 + \left(\sqrt{60}\right)^2 = h^2$ and hence $a^2 + 60 = h^2$. By the laws of algebra, $h^2 - a^2 = 60$ and so $(h+a)(h-a) = 60$. Since h and a are positive integers with $h > a$, $h + a$ and $h - a$ are also positive integers and $h + a > h - a$. Since $h + a$ and $h - a$ are factors of 60, the possibilities are:

$$h+a=60,\quad h-a=1; \qquad h+a=20,\quad h-a=3; \qquad h+a=12,\quad h-a=5;$$
$$h+a=30,\quad h-a=2; \qquad h+a=15,\quad h-a=4; \qquad h+a=10,\quad h-a=6.$$

When $h + a = 30$ and $h - a = 2$ then $h = 16$ and $a = 14$. This gives a right triangle with sides 14 and $\sqrt{60}$ and hypotenuse 16. When $h + a = 10$ and $h - a = 6$ then $h = 8$ and $a = 2$. This gives a right triangle with sides 2 and $\sqrt{60}$ and hypotenuse 8. In the other pairs of equations, the solutions for h and a are not integers. Thus there are two right triangles meeting our criteria: 14, $\sqrt{60}$, and 16, and 2, $\sqrt{60}$ and 8.

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18. Let $p = \log_M N$ so that $p = \log_N M$ as well. Then $M^p = N$ and $N^p = M$. Thus $M^{p^2} = (M^p)^p = N^p = M$. It follows that $p^2 = 1$ and hence $p = 1$ or $p = -1$. The choice $p = 1$ gives $M^1 = N$ and so $M = N$. Since this is not true, $p = -1$. Thus $M^{-1} = N$ and this means that $N = \dfrac{1}{M}$. Then $MN = M\left(\dfrac{1}{M}\right) = 1$.

19. Since $a+b+c=78$ and $a-b-c=40$, we may add equations to obtain $2a=118$ and hence $a=59$. It follows that $b+c=19$. The only solutions with b and c prime and $b>c$ is $b=17$ and $c=2$. Then $abc=(59)(17)(2)=2006$.

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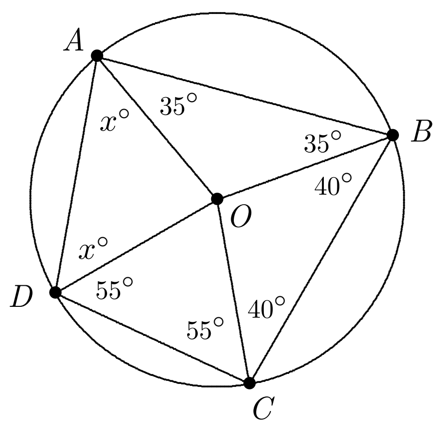
20. Let $\angle ODA = x^\circ$. Note that $OA=OB=OC=OD$ because OA, OB, OC, and OD are all radial segments. Then the triangles OAB, OBC, OCD, and ODA are all isosceles.



Thus $\angle OAB = \angle OBA = 35^\circ$, $\angle OBC = \angle OCB = 40^\circ$, $\angle OCD = \angle ODC = 55^\circ$, and $\angle ODA = \angle OAD = x^\circ$. Since the sum of the angles in any quadrilateral is 360°,

$$\begin{aligned} 2(35)+2(40)+2(55)+2x &= 360 \\ 260+2x &= 360 \\ 2x &= 100 \\ x &= 50 \end{aligned}$$

That is, $\angle ODA = 50^\circ$.

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21. We consider each expression in order. First, $(13+7)^5 = 20^5$, and any power of an even number is even. Second, $5(7^2-13)=5(49-13)=5(36)=180$. This is even. Third,

$$\frac{13}{7}+\frac{1}{2}-\frac{5}{14}=\frac{26}{14}+\frac{7}{14}-\frac{5}{14}=\frac{26+7-5}{14}=\frac{28}{14}=2.$$

This is even. Fourth, $\dfrac{13+7^2}{2}=\dfrac{13+49}{2}=\dfrac{62}{2}=31$. This is odd. Finally, $2(14+7)^{13}$ is even because 2 times any integer is even. The one and only odd integer among the choices given is $\dfrac{13+7^2}{2}$.

22. From the information given, $a - b = \pm 2$, $b - c = \pm 3$, and $c - d = \pm 4$. Since $a - d = (a - b) + (b - c) + (c - d)$, there are 8 possibilities for $a - d$:

$$
\begin{aligned}
a - d &= 2 + 3 + 4 = 9; & a - d &= -2 + 3 + 4 = 5;\\
a - d &= 2 + 3 - 4 = 1; & a - d &= -2 + 3 - 4 = -3;\\
a - d &= 2 - 3 + 4 = 3; & a - d &= -2 - 3 + 4 = -1;\\
a - d &= 2 - 3 - 4 = -5; & a - d &= -2 - 3 - 4 = -9.
\end{aligned}
$$

Then the possible values for $|a - d|$ are 1, 3, 5, and 9. The sum of these values is 18.

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23. Let O_1 and O_2 be the centres of the respective circles and let A and B be the points where the circles intersect. (See Figure 23(a) below.) Note that AO_1 and O_1O_2 are radii

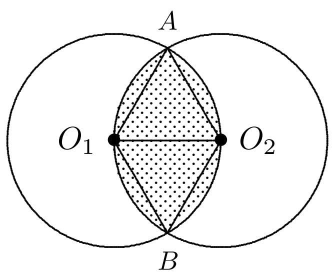

Fig. 23(a)

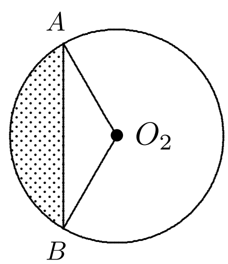

Fig. 23(b)

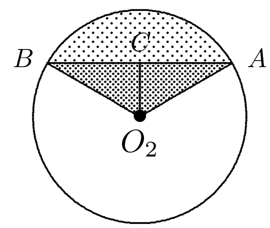

Fig. 23(c)

of the circle with centre O_1, and O_1O_2 and O_2A are radii of the circle with centre O_2. It follows that triangle AO_1O_2 is equilateral. Similarly, the triangle BO_1O_2 is equilateral. Thus $\angle AO_2B = 60^\circ + 60^\circ = 120^\circ$.

The shaded region in Figure 23(a) consists of two sectors as shaded in Figure 23(b). By rotating one sector 90°, this may be viewed as the upper portion of an "ice-cream cone" as in Figure 23(c). The area of the entire "ice-cream cone" is 1/3 of the area of the circle with centre O_2. (This is because $\angle AO_2B$ is 1/3 of an entire circle.) Thus its area is

$$\frac{1}{3}\,\pi(6^2) = \frac{1}{3}(36\pi) = 12\pi$$

To find the area of $\triangle BO_2A$, let C be the midpoint of BA. Then BO_2C and AO_2C are each 30-60-90 triangles. It follows that $BO_2 = AO_2 = 6$, $CO_2 = 3$, and $BC = AC = 3\sqrt{3}$. Then the area of BO_2A is

$$\frac{1}{2} \times BA \times CO_2 = \frac{1}{2}\big(6\sqrt{3}\,\big)(3) = 9\sqrt{3}.$$

Thus the area of one sector $\big($as in Figure 23(b)$\big)$ is $12\pi - 9\sqrt{3}$. The total shaded area $\big($as in Figure 23(a)$\big)$ is $24\pi - 18\sqrt{3}$.

24. If a number has remainder 4 when divided by 5 then its last digit is either 4 or 9. If the number is odd then its last digit must be 9. If the number is also less than 3568 then it is one of these numbers (counting backwards):

$$3559,\ 3549,\ 3539,\ 3529,\ 3519,\ 3509,\ \ldots$$

The largest number on this list which has remainder 2 when divided by 3 is 3539. The sum of its digits is 20.

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25. Let

$$\begin{aligned} S_1 &= \{1, 4, 7, 10, 13, 16, 19, 22, 25\}, \\ S_2 &= \{2, 5, 8, 11, 14, 17, 20, 23\}, \\ S_3 &= \{3, 6, 9, 12, 15, 18, 21, 24\}. \end{aligned}$$

Then S_1 consists of all the numbers in S with remainder 1 when divided by 3, S_2 consists of all the numbers with remainder 2, and S_3 consists of all the numbers divisible by 3. Note that the sum of any two numbers in S_1 is *not* divisible by 3. Similarly, the sum of any two numbers in S_2 is *not* divisible by 3. However, the sum of any number in S_1 and any number in S_2 *is* divisible by 3. Suppose T is a set with the greatest number of elements that can be chosen from S so that the sum of any two elements is *not* divisible by 3. Then T can contain all the elements in S_1 or all the elements in S_2, but it cannot contain elements from both subsets. Furthermore, suppose T contains all the elements in S_1. Then it can also contain one element from S_3. Similarly, if T contains all the elements in S_2 it may also contain one element from S_3. However, T cannot contain two elements from S_3. For example if T contains 6 and 15 then it contains $6 + 15 = 21$, and this is a multiple of 3. Then the possibilities are that:

a) T contains all elements in S_1 and one element from S_3, or

b) T contains all elements in S_2 and one element from S_3.

Since (a) gives a set with more elements, T contains 10 elements. For example, we could have $T = \{1, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$