THE UNIVERSITY OF THE WEST INDIES, MONA

Presents

The 2013 Jamaican Mathematical Olympiad

Qualifying Round

Solutions for Grades 7 and 8

1. To subtract 35 minutes from 7:10 pm we may subtract 10 minutes and then 25 minutes more. Subtracting 10 minutes gives 7:00 pm, and subtracting 25 minutes from 7:00 pm gives 6:35 pm. Daniel started at 6:35 pm.

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2. Observe that

$$\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}, \qquad \frac{1}{4}\times\frac{1}{4}=\frac{1}{16}, \qquad \frac{1}{4}-\frac{1}{4}=0, \quad \text{and} \quad \frac{1}{4}\div\frac{1}{4}=\frac{1}{4}\times\frac{4}{1}=1.$$

Then the largest of the five quantities given is $\frac{1}{4}\div\frac{1}{4}=1$.

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3. From the information given, Brenda gave half of her bananas to Anna and shared the other half between Catalina and herself. Since their half consisted of $3+4=7$ bananas, Brenda bought 14 bananas at the supermarket.

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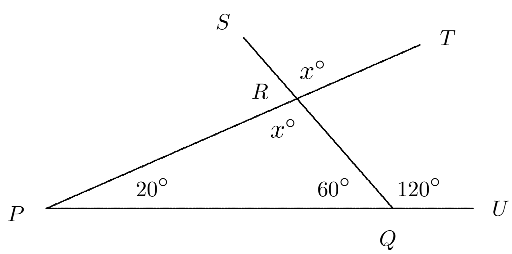
4. Let $\angle SRT = x^\circ$. By vertical angles, $\angle PRQ = x^\circ$ as well. It is given that $\angle RQU = 120$. Since $\angle RQP$ is its supplement, $\angle RQP = 60^\circ$. Since the sum of the angles in $\triangle PRQ$ is 180°, it follows that $x = 100$.



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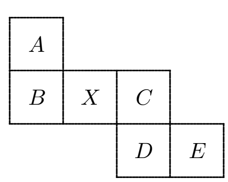
5. It is clear that the sides marked with B and C are adjacent to the one with X. Once the folds between these sides are made, it is also clear that the sides marked with A and D are also adjacent to the one with X. When the last fold is made, the side opposite the one with X will be marked with E.



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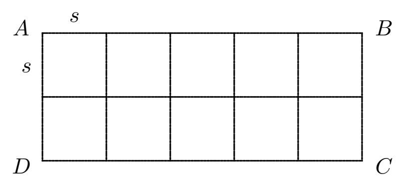
6. The given equation reduces to $0.0024 \times \square = 0.12$. Dividing both sides by 0.0024 gives $\square = 50$.

7. Let s be the side length (in centimetres) of one of the squares. Then $AB = 5s$, $BC = 2s$, $CD = 5s$, and $DA = 2s$. Thus $5s + 2s + 5s + 2s = 21$. Then $14s = 21$ and hence $s = 3/2$. The perimeter of each small square is $s + s + s + s = 4s$. Then the perimeter of each small square is $4\left(\frac{3}{2}\right) = 6$.



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8. It is given that 800 pennies have the value of 100 ducats, and 100 pennies have the value of 250 talars. Multiplying both of these latter quantities by 8, 800 pennies also have the value of 2000 talars. Then 100 ducats and 2000 talars have the same value. Dividing both of these quantities by 20, 5 ducats have the value of 100 talars.

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9. Equating the two diagonals (see Figure 9(a)), $8+x+y = 16+x+12$. Thus $8+x+y = x+28$. Subtracting x from both sides, $8 + y = 28$ and so $y = 20$. Substituting this value into the table (see Figure 9(b)), we see

16		y
	x	10
8		12

Fig. 9(a)

16		20
	x	10
8		12

Fig. 9(b)

from the third column that each row, column, and diagonal sums to 42. Then, from the upper left diagonal, $16 + x + 12 = 42$. This implies that $x = 14$. Finally, $x + y = 14 + 20 = 34$.

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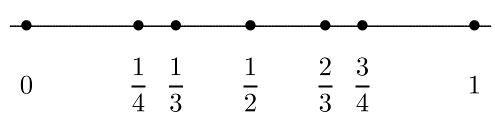
10. We may consider the segment of length 1 to be the part of the number line from 0 to 1, as shown below. The three points that divide the segment into four equal parts lie at 1/4, 1/2, and 3/4. These are marked

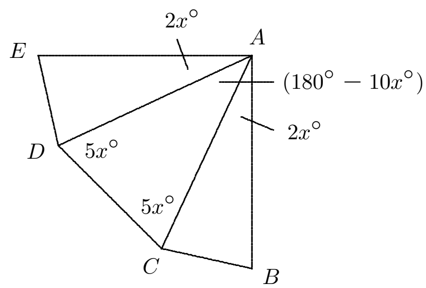


on the number line. The two points that divide the segment into three equal parts lie at 1/3 and 2/3. These are also marked on the number line. Then the six segments that these points produce are from 0 to 1/4, 1/4 to 1/3, 1/3 to 1/2, 1/2 to 2/3, 2/3 to 3/4, and 3/4 to 1. Their respective lengths are:

$$\frac{1}{4} - 0 = \frac{1}{4};\quad \frac{1}{4} - \frac{1}{3} = \frac{1}{12};\quad \frac{1}{2} - \frac{1}{3} = \frac{1}{6};\quad \frac{2}{3} - \frac{1}{2} = \frac{1}{6};\quad \frac{3}{4} - \frac{2}{3} = \frac{1}{12};\quad 1 - \frac{3}{4} = \frac{1}{4}.$$

There are three different numbers, 1/4, 1/12, and 1/6, that represent the lengths of these segments.

11. From the information given, $AD = AC$. Thus the triangle ADC is isosceles and $\angle ADC = \angle ACD = 5x°$. Since the sum of the angles in $\triangle ADC$ is $180°$, $\angle DAC = 180° - 10x°$. Since $\angle EAB$ is a right angle, $2x + (180 - 10x) + 2x = 90$. Then $180 - 6x = 90$ and it follows that $x = 15$.



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12. Suppose Bob and Ed shook hands. Then Alex and Bob would each have shaken hands with one person (Ed in both cases) and Ed would have shaken hands with two people. From the information given in the problem, they could not have shaken hands with anyone else. Then Carl and Doug would have to shake hands with two people, but they could not shake with Alex, Bob, or Ed. Since this is impossible, Bob definitely did not shake hands with Ed.

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13. Note that $4WXY - Y53Z = 2009$ if and only if $4WXY = Y53Z + 2009$. Thus the original problem (shown on the left below) is equivalent to the addition problem on the right. From the thousands column

$$\begin{array}{rcccc} & 4 & W & X & Y \\ - & Y & 5 & 3 & Z \\ \hline & 2 & 0 & 0 & 9 \end{array} \qquad\qquad \begin{array}{rcccc} & Y & 5 & 3 & Z \\ + & 2 & 0 & 0 & 9 \\ \hline & 4 & W & X & Y \end{array}$$

in the addition problem, it is clear that Y is either 1 or 2. Thus, from the ones column there, $Z + 9$ is either 11 or 12. In either case, a 1 will carry into the tens column. Then $1 + 3 + 0 = X$, and so $X = 4$. Also, $5 + 0 = W$, and so $W = 5$. Then $Y + 2 = 4$ and so $Y = 2$. Finally, from the ones column again, $Z + 9 = 12$ and so $Z = 3$. Thus $W \times X \times Y \times Z = 5 \times 4 \times 2 \times 3 = 120$.

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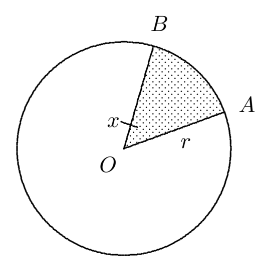
14. Let r be the radius of the circle and x the measure of $\angle AOB$. The area of the shaded sector is proportional to the measure of its central angle. That is,

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{x}{360}$$

Since the area of the sector is 15% of the area of the circle, we also have

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{15}{100} = \frac{3}{20}$$

Thus $\frac{x}{360} = \frac{3}{20}$. Cross multiplying, we obtain $20x = 1080$ and hence $x = 54$.



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15. Let a be Stephen's current age. In three years his age will be $a + 3$, and three years ago it was $a - 3$. Then $a + 3 = 3(a - 3)$. Thus

$$a + 3 = 3a - 9; \qquad 12 = 2a; \qquad 6 = a.$$

That is, Stephen is 6 years old. In four years his age will be 10, and four years ago he was 2. Then $10 = n(2)$ and hence $n = 5$.

16. Let the number of fish that Jose, Andres, and Miguel caught be J, A, and M, respectively. From the information given, $J = 3A$ and $J = 4M$. Since J is a multiple of both 3 and 4, it is a multiple of 12. Thus $J = 12n$ for some n. It follows that $A = 4n$ and $M = 3n$ as well. The total number of fish they caught was $12n + 4n + 3n = 19n$. Since they caught fewer than 100 fish, the most they could have caught was 95 fish, which occurs when $n = 5$. In this case, Jose would have caught 60 fish (and Andre and Miguel would have caught 20 and 15 fish, respectively).

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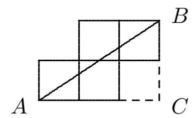
17. Extend some of the sides of the squares to a point C as shown in the figure below. By the Pythagorean theorem, $(AB)^2 = (AC)^2 + (CB)^2 = 3^2 + 2^2 = 9 + 4 = 13$. Then $AB = \sqrt{13}$.



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18. Suppose abc is a three-digit number with $a \times b \times c = 24$. Then $a, b, c \in \{1, 2, 3, 4, 6, 8\}$. If the largest digit of abc is 8 then the other two digits must be 3 and 1. There are six numbers of this type: 138, 183, 318, 381, 813, and 831. If the largest digit of abc is 6, its digits could be 6, 4, and 1, or 6, 2, and 2. There are six numbers of the first type: 146, 164, 416, 461, 614, and 641. There are three numbers of the second type: 226, 262, and 622. If the largest digit of abc is 4 then its digits must be 4, 3, and 2. There are six numbers of this type: 234, 243, 324, 342, 423, and 432. The largest digit of abc cannot be 3. This is because the other two digits would have to be less than or equal to 3 and have a product of 8. This is impossible. Similarly, the largest digit of abc cannot be 2 or 1. Therefore, there are $6 + 6 + 3 + 6 = 21$ numbers abc such that $a \times b \times c = 24$.

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19. Each 60 minutes the minute hand on a clock travels one full circle, which is 360°. Thus it travels at the rate of 6° per minute. The hour hand also travels one-twelfth of a full circle. This is 30°. Thus the hour

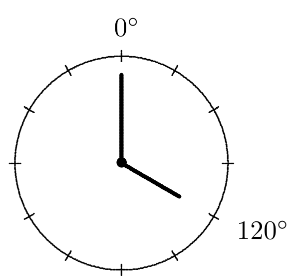

Fig. 19(a)

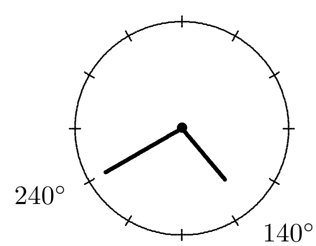

Fig. 19(b)

hand travels at the rate of 0.5° per minute. As shown in Figure 19(a), at 4:00 the minute hand is at 0° and the hour hand is at 120°. (We will use compass directions in this problem with north being at the 12.) After 40 minutes, the minute hand has traveled $40 \times 6° = 240°$ and ends at 240°. The hour hand has traveled $40 \times 0.5° = 20°$ and ends at 140°. Then the angle between the hands is 100°.

20. It is possible to form four triangles in such a way that none of them are right triangles and none are congruent to another. These are shown in Figures 20(a), 20(b), 20(c), and 20(d) below. All other triangles which can be formed are either right triangles or congruent to one of these.

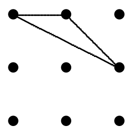

Fig. 20(a)

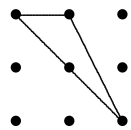

Fig. 20(b)

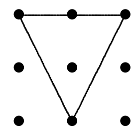

Fig. 20(c)

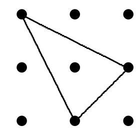

Fig. 20(d)

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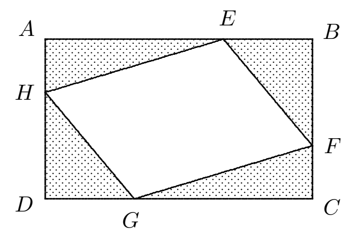
21. Let $AB = DC = \ell$ and $AD = BC = w$. Then $\ell w = S$. The area of $\triangle AEH$ is $\frac{1}{2}(AH)(AE)$. Since $AH = \frac{1}{3}(AD) = \frac{w}{3}$ and $AE = \frac{2}{3}(AB) = \frac{2\ell}{3}$, the area of $\triangle AEH$ is



$$\frac{1}{2}\left(\frac{w}{3}\right)\left(\frac{2\ell}{3}\right) = \frac{1}{9}(\ell w) = \frac{S}{9}.$$

Similarly, $EB = \frac{\ell}{3}$ and $BF = \frac{2w}{3}$. Then the area of $\triangle EBF$ is

$$\frac{1}{2}(EB)(BF) = \frac{1}{2}\left(\frac{\ell}{3}\right)\left(\frac{2w}{3}\right) = \frac{1}{9}(\ell w) = \frac{S}{9}.$$

In the same way, the areas of $\triangle HDG$ and $\triangle GCF$ are each $\frac{S}{9}$. Then the total shaded area is $\frac{S}{9}+\frac{S}{9}+\frac{S}{9}+\frac{S}{9} = \frac{4S}{9}$. It follows that the area of the parallelogram $EFGH$ is

$$S - \frac{4S}{9} = \frac{9S}{9} - \frac{4S}{9} = \frac{5S}{9}.$$

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22. Suppose m is a perfect square. Then $m = n^2$ for some positive integer n. Let n have the prime factorization $n = p^a q^b \cdots r^c$, where p, q, …, r are distinct prime numbers and a, b, …, c are positive integers. Then

$$m = n^2 = (p^a q^b \cdots r^c)(p^a q^b \cdots r^c) = p^{2a} q^{2b} \cdots r^{2c}$$

Thus each prime number in the factorization of m has an even exponent. On the other hand, suppose m is a number and each of its prime factors has an even exponent. Then m is a perfect square. Indeed, if each exponent ie even then each is a multiple of 2. Thus m can be written as

$$m = p^{2a} q^{2b} \cdots r^{2c}$$

for some prime numbers p, q, …, r and some positive integers a, b, …, c. Let $n = p^a q^b \cdots r^c$. Then n is an integer and $n^2 = m$, as one may verify. Thus m is a perfect square. We may therefore conclude that a number m is a perfect square if and only if every exponent in its prime factorization is a multiple of 2. Reasoning in the same way, we may conclude that a number m is a perfect cube if and only if every exponent in its prime factorization is a multiple of 3.

The prime factorization of 360 is $2^3 3^2 5$. Let x be the smallest number such that $360x$ is a perfect square. Clearly, $360x$ must have an even numbers of 2's, 3's and 5's in its prime factorization. The smallest value for x that makes $360x$ a perfect square is $x = 2(5) = 10$. Let y be the smallest number such that $360y$ is a perfect cube. The number of 2's, 3's and 5's in the prime factorization of $360y$ must all be multiples of 3. The smallest value for y that makes $360y$ a perfect cube is $y = 3(5^2) = 75$. Thus, $x + y = 10 + 75 = 85$.

23. A number n between 1 and 1000 could have one, two, or three digits. If n has one digit it could be 2, 3, 4, ..., or 9. There are 8 possibilities in all. If n has two digits, it has the form ab, where $a \in \{2, 3, 4, \ldots, 9\}$ and $b \in \{0, 2, 3, 4, \ldots, 9\}$. There are 8 possibilities for a and 9 for b. Then there are $8 \times 9 = 72$ possibilities for ab in all. If n has three digits, it has the form abc, where $a \in \{2, 3, 4, \ldots, 9\}$, $b \in \{0, 2, 3, 4, \ldots, 9\}$, and $c \in \{0, 2, 3, 4, \ldots, 9\}$. There are 8 possibilities for a and 9 each for b and c. Then there are $8 \times 9 \times 9 = 648$ possibilities for abc in all. Considering all three cases, there are $8 + 72 + 648 = 728$ numbers between 1 and 1000 that do not contain the digit 1.

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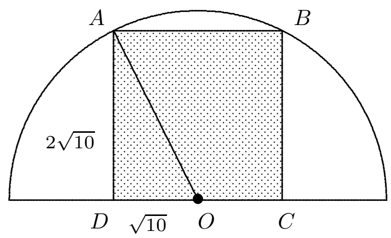
24. Let the square be $ABCD$ and let O be the centre of the semicircle, as shown below. Since the square has area 40, each side has length $\sqrt{40} = 2\sqrt{10}$. It follows that $AD = 2\sqrt{10}$ and $OD = \sqrt{10}$. Note that OA



is a radius of the semicircle. By the Pythagorean theorem,

$$(OA)^2 = (AD)^2 + (OD)^2 = \left(2\sqrt{10}\right)^2 + \left(\sqrt{10}\right)^2 = 40 + 10 = 50.$$

Then $OA = \sqrt{50} = 5\sqrt{2}$. Thus the area of the semicircle is $\frac{1}{2}\pi(OA)^2 = \frac{\pi}{2}\left(5\sqrt{2}\right)^2 = \frac{\pi}{2}(50) = 25\pi$.

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25. First, observe that

$$\frac{1}{5^{2012}} = \frac{1}{5^{2012}} \times \frac{2^{2012}}{2^{2012}} = \frac{2^{2012}}{10^{2012}}$$

In decimal notation, this is the number 2^{2012} with the decimal point moved 2012 places to the left. The last digit of this expression is the same as the last digit of 2^{2012} itself. The powers of 2 are 2, 4, 8, 16, 32, 64, 128 256, 512, etc. Their last digits are 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, ... This sequence repeats in cycles of four. Thus the 4th, 8th, 12th, 16th, ..., 2012th terms are all 6. Then last digit of $1/5^{2012}$ is 6.