THE UNIVERSITY OF THE WEST INDIES, MONA

Presents

The 2013 Jamaican Mathematical Olympiad

Qualifying Round

Solutions for Grades 9, 10, and 11

1. Using ordinary laws of algebra, $2-(-4)=2+4=6$; $(-2)\times(-3)=-(-6)=6$; $2-8=-6$; $0-(-6)=0+6=6$; $(-12)\div(-2)=-(-6)=6$. Then exactly one of these expressions, the third one, is *not* equal to 6.

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2. Observe that $(-1)^1=-1$, $(-1)^2=1$, $(-1)^3=-1$, $(-1)^4=1$, and so on. In general, -1 raised to any odd power is -1 and -1 raised to any even power is 1. Then the given sum is

$$(-1)+1+(-1)+1+(-1)+1+\cdots+(-1)+1.$$

This is a sum of 2012 terms, and 1006 of them are -1 while the other 1006 are 1. Then the total sum is $-1006+1006=0$.

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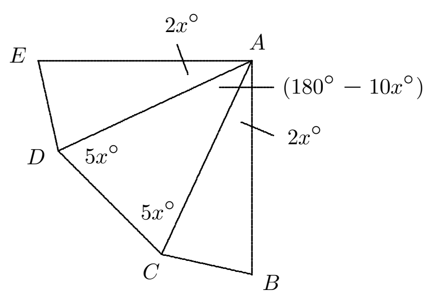
3. From the information given, $AD=AC$. Thus the triangle ADC is isosceles and $\angle ADC=\angle ACD=5x°$. Since the sum of the angles in $\triangle ADC$ is 180°, $\angle DAC=180°-10x°$. Since $\angle EAB$ is a right angle, $2x+(180-10x)+2x=90$. Then $180-6x=90$ and it follows that $x=15$.



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4. From the information given, Western High School has 430 boys and 570 girls. Since 313 boys do not ride the bus, 117 of them do. Since 250 students ride the bus to school, this means that 133 of them are girls.

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5. Since P is equal to Q plus 40% of Q, $P=Q+0.40Q=(1+0.4)Q=1.4Q$. Then

$$\frac{P}{Q}=\frac{1.4Q}{Q}=\frac{1.4}{1}=\frac{14}{10}=\frac{7}{5}$$

That is, $P:Q=7:5$.

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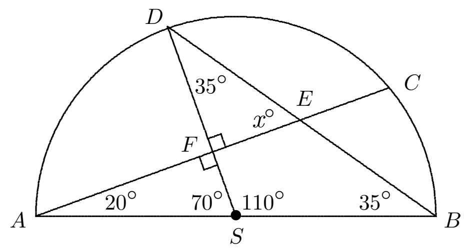
6. Let F be the point where AC and DS intersect. From the information given, $\angle FAS=20°$ and $\angle AFS=90°$. Since the sum of the angles in $\triangle AFS$ is 180°, $\angle ASF=70°$. Since $\angle DSB$ is the supplement



of $\angle DSA = \angle FSA$, we have $\angle DSB = 110°$. The triangle DSB is isosceles with $SB = SD$. Since the sum of the angles in $\triangle DSB$ is $180°$, it follows that $\angle SDB = \angle SBD = 35°$. Since $\angle DFE = 90°$ and the sum of the angles in $\triangle DFE$ is $180°$, we have $x = 55$.

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7. Since $9 = 3^2$, we have $9^{30} = (3^2)^{30} = 3^{60}$. If $3^k = 9^{30}$ then $3^k = 3^{60}$. Thus $k = 60$.

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8. If $p = 2$ then $p^4 + 1 = 2^4 + 1 = 17$. This is prime. Suppose now that p is a prime number greater than 2. Then p must be odd. (If p were even it would have 1, 2, and p as factors and would not be prime.) Then p^4 is odd and hence $p^4 + 1$ is even. Since $p^4 + 1$ is greater than 2, it is cannot be prime. Then $p = 2$ is the only prime number such that $p^4 + 1$ is also prime.

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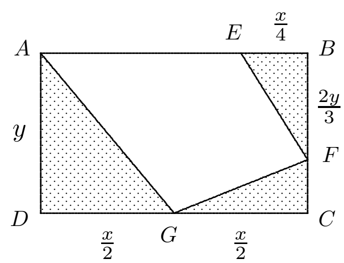
9. Suppose $AB = BC = x$ and $AD = BC = y$. Then the area of the rectangle $ABCD$ is xy, and $DG = GC = x/2$, $FC = y/3$, $FB = 2y/3$, and $EB = x/4$. The area of $\triangle ADG$ is $\frac{1}{2}(y)\left(\frac{x}{2}\right) = \frac{xy}{4}$. The area of $\triangle GCF$ is



$\frac{1}{2}\left(\frac{x}{2}\right)\left(\frac{y}{3}\right) = \frac{xy}{12}$. The area of $\triangle FBE$ is $\frac{1}{2}\left(\frac{2y}{3}\right)\left(\frac{x}{4}\right) = \frac{xy}{12}$. Then the total shaded area is

$$\frac{xy}{4} + \frac{xy}{12} + \frac{xy}{12} = \frac{3xy}{12} + \frac{xy}{12} + \frac{xy}{12} = \frac{5xy}{12}$$

Since the area of the rectangle is xy, the area of the quadrilateral $EFGA$ is $xy - \frac{5xy}{12} = \frac{7xy}{12}$. Then the ratio of area of $EFGA$ to that of $ABCD$ is

$$\frac{7xy/12}{xy} = \frac{7}{12}$$

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10. Multiply both equations together to obtain $(x^2yz^3)(xy^2) = (7^3)(7^9)$ and hence $x^3y^3z^3 = 7^{12}$. Taking the cube root of both sides (or raising both sides to the 1/3 power), $xyz = 7^4$.

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11. Suppose Jordan bought p paperbacks and h hardbound books. Then $500p + 700h = 10{,}100$. Dividing throughout by 100, $5p + 7h = 101$. Thus $5p = 101 - 7h$. Since $5p$ is obviously a positive multiple of 5, so is $101 - 7h$. This happens when $h = 3$, 8, and 13. In the first case $h = 3$ and $p = 16$, and Jordan would have bought 19 books in all. In the second case $h = 8$ and $p = 9$, and he would have bought 17 books in all. In

the last case $h = 13$ and $p = 2$, and he would have bought 15 books in all. Then the smallest number of books Jordan could have bought is 15.

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12. If $x^2 + y^2 = 50$ then $-8 < x < 8$ and $-8 < y < 8$. We list all possible grid points below. The first column has all points where x is an integer and the second one has all points where y is an integer,

$$\begin{array}{ll}
(0, \pm\sqrt{50}) & (\pm\sqrt{50}, 0) \\
(\pm 1, \pm 7) & (\pm 7, \pm 1) \\
(\pm 2, \pm\sqrt{46}) & (\pm\sqrt{46}, \pm 2) \\
(\pm 3, \pm\sqrt{41}) & (\pm\sqrt{41}, \pm 3) \\
(\pm 4, \pm\sqrt{34}) & (\pm\sqrt{34}, \pm 4) \\
(\pm 5, \pm 5) & (\pm 5, \pm 5) \\
(\pm 6, \pm\sqrt{14}) & (\pm\sqrt{14}, \pm 6) \\
(\pm 7, \pm 1) & (\pm 1, \pm 7)
\end{array}$$

The first column contains 30 points and the second one contains 30 points. However, the pairs $(\pm 1, \pm 7)$, $(\pm 5, \pm 5)$, and $(\pm 7, \pm 1)$ occur in each column. This gives 12 duplicate points in all. Then there are $60 - 12 = 48$ grid points in all.

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13. Any number n meeting the stated criteria has the form $2ab8$, where a and b are digits. Furthermore, the fact that n is a multiple of 3 means that the sum of its digits, $2 + a + b + 8 = a + b + 10$, is a multiple of 3. We list the possible choices for a and b below.

$$\begin{array}{llll}
a = 0; & b = 2, 5, 8; & a = 5; & b = 0, 3, 6, 9; \\
a = 1; & b = 1, 4, 7; & a = 6; & b = 2, 5, 8; \\
a = 2; & b = 0, 3, 6, 9; & a = 7; & b = 1, 4, 7; \\
a = 3; & b = 2, 5, 8; & a = 8; & b = 0, 3, 6, 9; \\
a = 4; & b = 1, 4, 7; & a = 9; & b = 2, 5, 8;
\end{array}$$

There are 33 possible numbers in all.

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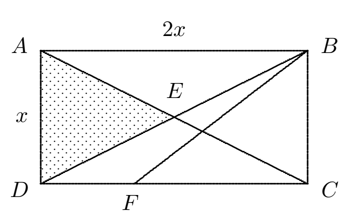
14. The area of the pentagon $BAEDC$ is the area of the rectangle $ABCD$ minus the area of the triangle ADE. The area of $ABCD$ is $2x^2$. Triangle ADE has base AD and its height is the perpendicular distance



from E to AD. Since its base and height are each x, its area is $\frac{1}{2}x^2$. Then the area of the pentagon $BAEDC$ is $2x^2 - \frac{1}{2}x^2 = \frac{3}{2}x^2$. Given that the triangle FCB has half this area, its area is $\frac{3}{4}x^2$. But its area is also $\frac{1}{2}(FC)(BC) = \frac{1}{2}(FC)x = \frac{1}{2}x(FC)$. Then $\frac{1}{2}x(FC) = \frac{3}{4}x^2$ and it follows that $FC = \frac{3}{2}x$.

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15. When Jacob poured 2/3 of the juice from a pitcher into a jar, 1/3 of it remained in the pitcher. Of the juice in the jar, he drank 5/8 of it and poured 1/8 of it into a glass for later. This accounts for $\frac{5}{8} + \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$

of it. Thus he poured $\frac{1}{4}$ of it back into the pitcher. At the end, the amount of juice in the pitcher, as a fraction of the original, was

$$\frac{1}{3}+\frac{1}{4}\left(\frac{2}{3}\right)=\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}.$$

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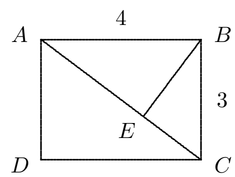
16. The triangles ABC, AEB, and BEC are all right triangles with right angles at B, E, and E, respectively. By the Pythagorean theorem, $AC = 5$. Since ABC and AEB share an angle at A, their third angles are



also equal. (The sum of the angles in any triangle is 180°). Thus ABC and AEB are similar. Similarly, since ABC and BCE share an angle at C, their third angles are also equal. Thus ABC and BCE are similar. Then the triangles ABC, AEB, and BEC are all similar to each other. Since corresponding sides of similar triangles are proportional,

$$\frac{EC}{BC}=\frac{BC}{AC};\qquad \frac{EC}{3}=\frac{3}{5};\qquad EC=\frac{9}{5}.$$

In the same way,

$$\frac{AE}{AB}=\frac{AB}{AC};\qquad \frac{AE}{4}=\frac{4}{5};\qquad AE=\frac{16}{5}.$$

Finally,

$$\frac{EC}{EB}=\frac{EB}{EA};\qquad \frac{9/5}{EB}=\frac{EB}{16/5};\qquad (EB)^2=\frac{144}{25};\qquad EB=\frac{12}{5}=2.4$$

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17. On five of Mark's cards the one's digit of the number shown is 3, and on the other five it is 8. In order to have numbers on the cards adding up to 100, their one's digits must add up to a number ending in 0. If the last digits are 3, 3, 3, 3, and 8 their sum is 20. In this case five cards would be needed. If the last digits are 3, 3, 8, 8, and 8 their sum is 30. In this case five cards would also be needed. It is not possible to use four or fewer cards and have the sum of the one's digits end in 0 (try it). So if it is possible at all to have the numbers on the cards sum to 100, at least five of them are needed.

In fact, it is possible to use exactly five cards. Note that any of these combinations sum to 100:

$$3,13,23,33,28;\qquad 3,13,23,53,8;\qquad 13,33,8,18,28;\qquad 3,23,8,18,48;\qquad 3,13,8,28,48.$$

So the minimum number of cards Mark has to choose is five.

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18. Subtracting the second equation from the first one gives $ab - ac = 44 - 23$, or $a(b - c) = 21$. Since a and $b - c$ are integers and a is positive, there are four possibilities to consider: $a = 1$, $b - c = 21$; or $a = 3$, $b - c = 7$; or $a = 7$, $b - c = 3$; or $a = 21$, $b - c = 1$. We consider each one in turn.

Case 1. If $a = 1$ then the second of the original equations reduces to $c(1 + b) = 23$. The only possibilities are $c = 23$, $1 + b = 1$, or $c = 1$, $1 + b = 23$. In the first instance we would have $b = 0$, which is impossible. The second one gives $b = 22$, $c = 1$ (and $a = 1$). This is a valid solution, as one may verify.

Case 2. If $a = 3$ then the second of the original equations reduces to $c(3 + b) = 23$. The only possibilities are $c = 23$, $3 + b = 1$, or $c = 1$, $3 + b = 23$. In the first instance we would have $b = -2$, which is impossible. The second one gives $b = 20$, $c = 1$ (and $a = 3$). However, this is not a valid solution.

Case 3. If $a = 7$ then the second of the original equations reduces to $c(7 + b) = 23$. The only possibilities are $c = 23$, $7 + b = 1$, or $c = 1$, $7 + b = 23$. In the first instance we would have $b = -6$, which is impossible. The second one gives $b = 16$, $c = 1$ (and $a = 7$). However, this is not a valid solution.

Case 4. If $a = 21$ then the second of the original equations reduces to $c(21 + b) = 23$. The only possibilities are $c = 23$, $21+b = 1$, or $c = 1$, $21+b = 23$. In the first instance we would have $b = -20$, which is impossible. The second one gives $b = 2$, $c = 1$ (and $a = 21$). This is a valid solution.

In summary, there are exactly two triples (a, b, c) of positive integers solving the original system. These are $(1, 22, 1)$ and $(21, 2, 1)$.

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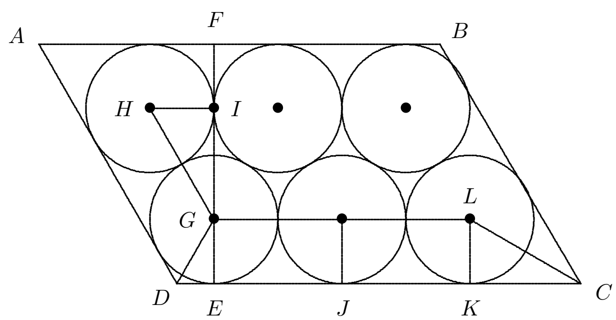
19. Mark the centres of the six circles and let H, G, and L be three of them, as shown in the figure below. Let EF be perpendicular to DC and pass through G. Then EF also passes through I, the point of tangency between the two circles as shown. The area of the parallelogram is $(DC)(EF)$. To find DC and EF, note



that whenever three circles are mutually tangent their centres form an equilateral triangle. It follows that the angles in the parallelogram $ABCD$ are $60°$ (at A and C) and $120°$ (at B and D). Also, GHI, GDE, and CLK are all 30-60-90 triangles. Since each circle has radius 3, $GE = LK = 3$ and $GL = EK = 12$. Also, $DE = \sqrt{3}$ and $KC = 3\sqrt{3}$. (In any 30-60-90 triangle we have, in ratio, *shortest side* : *middle side* : *longest side* $= 1 : \sqrt{3} : 2$.) Then

$$DC = \sqrt{3} + 12 + 3\sqrt{3} = 12 + 4\sqrt{3} = 4(3 + \sqrt{3})$$

Similarly, $EG = 3$, $GI = 3\sqrt{3}$, and $IF = 3$. Thus $EF = 3 + 3\sqrt{3} + 3 = 6 + 3\sqrt{3} = 3(2 + \sqrt{3})$. Then

$$\begin{aligned}(DC)(EF) &= 4(3 + \sqrt{3})3(2 + \sqrt{3}) \\ &= 12(6 + 3\sqrt{3} + 2\sqrt{3} + 3) = 12(9 + 5\sqrt{3})\end{aligned}$$

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20. Since $n!$ contains 13 as a factor, $n \geq 13$. Note that

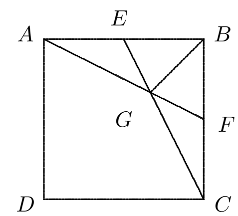
$$\begin{aligned}13! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \\ &= 2 \times 3 \times 2^2 \times 5 \times (2 \times 3) \times 7 \times 2^3 \times 3^2 \times (2 \times 5) \times 11 \times (2^2 \times 3) \times 13 \\ &= 2^{10} \times 3^5 \times 5^2 \times 7 \times 11 \times 13\end{aligned}$$

Since $14 = 2 \times 7$, $15 = 3 \times 5$, and $16 = 2^4$,

$$\begin{aligned}16! &= 13! \times 14 \times 15 \times 16 = \left(2^{10} \times 3^5 \times 5^2 \times 7 \times 11 \times 13\right) \times (2 \times 7) \times (3 \times 5) \times (2^4) \\ &= 2^{15} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13\end{aligned}$$

This is exactly the same as the $n!$ in the statement of the problem. Thus $n = 16$.

21. Let the square $ABCD$ have side length s and area s^2. Then EBC is a right triangle with $EB = \frac{1}{2}s$ and $BC = s$. Its area is $\frac{1}{2}\left(\frac{1}{2}s\right)(s) = \frac{1}{4}s^2$. Join B and G with a line segment as shown. We prove that the area of $\triangle GBC$ is two-thirds the area of $\triangle EBC$. It is shown below that $\angle GBE = \angle GBF = 45°$. Assuming so for now, the triangles GBE and GBF are congruent by the side-angle-side theorem. Indeed, $EB = FB = \frac{1}{2}s$, $\angle EBG = \angle FBG$, and the side BG is common to both triangles. Then $\triangle GBE$ and $\triangle GBF$ have the same area. Furthermore, $\triangle GBF$ and $\triangle GFC$ have the same area. This is because they have equal bases (BF and FC, respectively) and equal heights (the perpendicular distance from G to BC). Thus the triangle EBF is divided into three smaller triangles, GEB, GBF, and GFC, with equal area. It follows that the area of $\triangle GBC$ is two-thirds that of $\triangle EBC$.



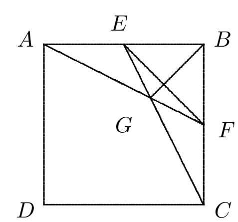
Since the area of $\triangle EBC$ is $\frac{1}{4}s^2$, the area of $\triangle GBC$ is therefore $\frac{2}{3}\left(\frac{1}{4}s^2\right) = \frac{1}{6}s^2$. Reasoning in the same way, the area of $\triangle GBA$ is also $\frac{1}{6}s^2$. Then the area of the quadrilateral $ABCG$ is

$$\frac{1}{6}s^2 + \frac{1}{6}s^2 = \frac{1}{3}s^2.$$

Since the area of the square $ABCD$ is s^2, The area of the quadrilateral $AGCD$ is $s^2 - \frac{1}{3}s^2 = \frac{2}{3}s^2$. Finally, the ratio of the area of $AGCD$ to the area of $ABCD$ is

$$\frac{2s^2/3}{s^2} = \frac{2}{3}$$

To show that $\angle GBE = \angle GBF = 45°$, join E to F with a line segment. Then $\triangle EBF$ is isosceles with $EB = FB$. Since $\angle EBF = 90°$ and the sum of the angles in $\triangle EBF$ is 180°, $\angle EFB = \angle FEB = 45°$. Also, $\triangle ABF$ and $\triangle CBE$ are congruent by the side-angle-side theorem. Indeed, $EB = FB$, $\angle ABF = \angle CBE$, and $BC = BA$. Thus $\angle CEB = \angle AFB$. Subtracting 45° from both sides, $\angle CEF = \angle AFE$. This implies that the triangle GEF is isosceles with $GE = GF$. Then the triangles BEG and BFG are congruent by the side-side-side theorem. Thus $\angle GBE = \angle GBF$, and each one is one-half of a 90° angle.



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22. The multiples of 17 and 23 which are two-digit numbers are:

17	23
34	46
51	69
68	92
85	

Suppose a number with 2012 digits is primo. If one of the digits is 8 then the only digits that can follow are 5, 1, and 7 (in that order). After that, there can be no more digits. This means that the digit 8 can occur only as the 2009th, 2010th, 2011th, or 2012th digit.

Suppose now that n has 2012 digits and is primo. Its first digit cannot be 1. Otherwise, its second digit is 7 and there is no possible third digit. Similarly, its first digit cannot be 5. Otherwise its first three digits are 517 and there is no possible fourth digit. Similarly, its first digit cannot be 7 (no second digit is possible) and its first digit cannot be 8 (no 8 can come before the 2009th digit). Then the first digit of n must be 2, 3, 4, 6, or 9.

Suppose the first digit of n is 2. Then its first four digits are 2346. Since the next digit cannot be 8, it must be 9 and the one following must be 2. Thus the number has the form $234692346923469\cdots2346xyz$.

In other words, the digits repeat in cycles of five and 6 occurs as the 4th, 9th, 14th 19th, ..., 2004th, and 2009th digits. After that, x could be 8 or 9. This gives the two possible numbers

$$234692346923469\cdots2346851$$
$$234692346923469\cdots2346923$$

Reasoning in the same way, if the first digit of n is 3 it has the form $346923469234692\cdots346xyzw$. The digits occur in cycles of 5 again but this time 6 occurs as the 3rd, 8th, 13th 18th, ..., 2003rd, and 2008th digits. Once again, x could be 8 or 9. This gives two more possible numbers:

$$346923469234692\cdots3468517$$
$$346923469234692\cdots3469234$$

If the first digit of n is 4 it has the form $469234692346923\cdots46xyzwu$. In this case, x cannot be 8 because it falls in the 2008th position. Thus x is 9 and the only possible number is

$$469234692346923\cdots4692346.$$

If the first digit of n is 6 it has the form $692346923469234\cdots692346x$. In this case, x could be 8 or 9 again. This gives two more possible numbers:

$$692346923469234\cdots6923468$$
$$692346923469234\cdots6923469$$

If the first digit of n is 9 it has the form $923469234692346\cdots92346xy$. In this case, x could be 8 or 9 again. This gives two more possible numbers:

$$923469234692346\cdots9234685$$
$$923469234692346\cdots9234692$$

All possibilities have now been considered. There are nine different numbers which are primo and have 2012 digits.

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23. It is given that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$. Finding common denominators,

$$\frac{yz}{xyz}+\frac{xz}{xyz}+\frac{xy}{xyz}=2; \qquad \frac{xy+xz+yz}{xyz}=2; \qquad \frac{xy+xz+yz}{4}=2; \qquad xy+xz+yz=8.$$

Also $(x+y+z)^2=(5)^2=25$. But

$$\begin{aligned}(x+y+z)^2 &= x^2+xy+xz+xy+y^2+yz+xz+yz+z^2\\ &= x^2+y^2+z^2+2(xy+xz+yz)\\ &= x^2+y^2+z^2+2(8)\\ &= x^2+y^2+z^2+16\end{aligned}$$

Then $x^2+y^2+z^2+16=25$ and so $x^2+y^2+z^2=9$.

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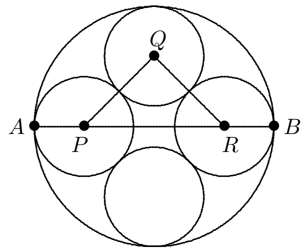
24. Let A and B be the endpoints of a diameter and P, Q, and R the centres of small circles as shown in the figure below. Let r be the radius of a small circle. Then each small circle has area πr^2, and the sum of



their areas is $4\pi r^2$. To find the area of the large, circumscribing circle we first determine AB. We know that $AP = r$, $PQ = 2r$, $QR = 2r$, and and $RB = r$. Also, the triangle PQR is a right triangle with a right angle at Q. By the Pythagorean theorem, $(PQ)^2 + (QR)^2 = (PR)^2$. Thus

$$(2r)^2 + (2r)^2 = (PR)^2; \qquad 4r^2 + 4r^2 = (PR)^2; \qquad 8r^2 = (PR)^2; \qquad PR = \sqrt{8r^2} = 2\sqrt{2}r.$$

Then $AB = AP + PR + RB = r + 2\sqrt{2}r + r = 2(1+\sqrt{2})r$. It follows that the radius of the circumscribing circle is $(1+\sqrt{2})r$, and its area is

$$\pi(1+\sqrt{2})^2 r^2 = \pi(1+2\sqrt{2}+2)r^2 = \pi(3+2\sqrt{2})r^2.$$

Then the ratio of the sum of the areas of the four smaller circles to the area of the larger one is

$$\frac{4\pi r^2}{\pi(3+2\sqrt{2})r^2} = \frac{4}{3+2\sqrt{2}}\,\frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{4(3-2\sqrt{2})}{9-8} = 4(3-2\sqrt{2})$$

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25. Suppose all the digits in a number are either 1 or 3 and they sum to 10. Then there can be ten 1s, or seven 1s and one 3, or four 1s and two 3s, or one 1 and three 3s. We consider each possibility in turn.

Ten 1s. If a number has ten digits and they are all 1, it must be 1,111,111,111. There is only one possibility in this case.

Seven 1s and One 3. Suppose a number has eight digits and seven of them are equal to 1 and the other is 3. Then the number must be one of these:

$$31,111,111, \qquad 13,111,111, \qquad 11,311,111, \quad \cdots \quad 11,111,131, \qquad 11,111,113$$

There are eight possibilities in this case. In fact, this is exactly "eight choose one".

Four 1s and Two 3s. Suppose a number has six digits and four of them are equal to 1 and the others are 3. Then the number must be one of these:

331,111,	133,111,	113,311,	111,331,	111,133,
313,111,	131,311,	113,131,	111,313,	
311,311,	131,131,	113,113,		
311,131,	131,113,			
311,113				

There are 15 possibilities in all. In fact, this is exactly "six choose two".

One 1 and Three 3s. Suppose a number has four digits and one of them is equal to 1 and the others are 3. Then the number must be one of these:

$$1,333, \qquad 3,133, \qquad 3,313, \qquad 3,331.$$

There are four possibilities in all. In fact, this is exactly "four choose one".

Putting all of the cases together, there are $1 + 8 + 15 + 4 = 28$ numbers with the property that each of their digits is 1 or 3 and the sum of the digits is 10.