

The 2014 Junior Mathematical Olympiad

Practice Problems Set 2

Logic, Pattern, and Counting Solutions

1. There are three small rectangles of the type shown in Figure 1(a) below. There are two medium-sized rectangles (each made of two smaller ones) of the type shown in Figure 1(b) below. There is one large rectangle as shown in Figure 1(c) below. We may find $3 + 2 + 1 = 6$ rectangles in the diagram all together.

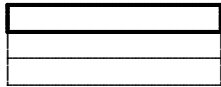


Fig. 1(a)

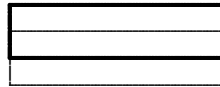


Fig. 1(b)

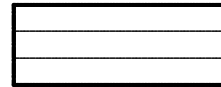


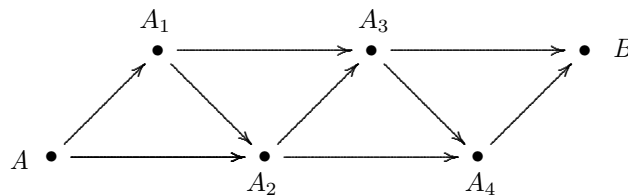
Fig. 1(c)

2. From the information given, the 1st floor of Shawana's building contains shops, the 2nd floor contains Apartments 1, 2, and 3, and so on. We may produce the table below. Apartment 17 is on the 7th Floor.

7th Floor	Apartments 16, 17, 18
6th Floor	Apartments 13, 14, 15
5th Floor	Apartments 10, 11, 12
4th Floor	Apartments 7, 8, 9
3rd Floor	Apartments 4, 5, 6
2nd Floor	Apartments 1, 2, 3
1st Floor	Shops

3. Suppose the three-digit number is abc . There are 2 possibilities for a : 3 or 7. (We cannot have $a = 0$; a number such as 073 is not a three-digit number.) For each one, there are two possibilities for b . Once these are chosen, there is only one possibility left for c . So, there are $2 \times 2 \times 1 = 4$ such numbers in all. They are 307, 370, 703, and 730.

4. Let $A_1, A_2, A_3,$ and A_4 be the points shown in the figure below. There is exactly 1 path from A_4 to B , and there are 2 paths from A_3 to B . Starting from A_2 , one can proceed to A_3 and choose from 2 paths to B ,



or proceed directly to A_4 and take 1 path to B . So, there are $2 + 1 = 3$ paths from A_2 to B . From A_1 , one can proceed to A_2 and choose from 3 paths to B , or proceed directly to A_3 and choose from 2 more paths to B . There are $3 + 2 = 5$ paths from A_1 to B . Finally, starting from A one can proceed to A_1 and choose from 5 paths to B , or proceed directly to A_2 and choose from 3 more paths to B . There are a total of $5 + 3 = 8$ paths from A to B .

5. The multiples of 7 between 100 and 1000 are 105, 112, 119, 126, \dots , 994. These are the numbers $7(15)$, $7(16)$, $7(17)$, $7(18)$, \dots , $7(142)$. In other words, these are the numbers 15 through 142 multiplied by 7. There are $142 - 15 + 1 = 128$ such numbers in all.

6. Each of the letters A and B stands for one of the numbers 0, 1, 2, \dots , 9. In the units column, when B is added to 6 the sum is a number ending with 8. The only possibility is $B = 2$. In the tens column, when A is added to 9 the sum is a number ending with 2. The only possibility is $A = 3$. The number AB is 32.

$$\begin{array}{r} 296 \\ + AB \\ \hline AB8 \end{array}$$

The first digit of an uphill number cannot be 9 because there is no digit greater than 9 that could be its second digit. So, there are $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ uphill numbers in all.

7. If there are 2 or more red balls in the box then there would be 14 or more white balls in the box. There would be at least 16 balls in the box in all. Since this is not so, there is only one red ball in the box. (There has to be at least one of them in the box.) Then there are 7 white balls in the box. This means that there are also 7 black balls in the box.

8. Suppose the teams are A , B , C , and D . Since each team played every other team once, the matches were $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$. (Here, $\{A, B\}$ means that the teams A and B play each other, and so on.) There were 6 matches in all. The number of ties had to be 0, 1, 2, 3, 4, 5, or 6. Also, the total number of match points awarded was $5 + 3 + 3 + 2 = 13$. When a match had a winner a total of 3 match points (3 for the winners and none for the losers) were awarded. When a match was tied a total of 2 match points (1 for each team) were awarded. Suppose all six matches were ties. Then a total of 12 match points would be awarded. But this was not so. Suppose 5 matches were ties (and one match had a winner). A total of $1 \times 3 + 5 \times 2 = 13$ match points would be awarded. This was in fact so. Suppose 4 matches were ties (and 2 matches had a winner). A total of $2 \times 3 + 4 \times 2 = 14$ match points would be awarded. This was not so. Similarly, if the number of ties was 3, 2, 1, or 0, the number of match points awarded would have been 15, 16, 17, or 18, respectively. So, the only possibility is that there were 5 ties in the tournament.

9. There are three kinds of rolling numbers:

- a) ones with the first two digits the same and the third one different (for example, 775);
- b) ones with the last two digits the same and the first one different (for example, 144);
- c) ones with all three digits the same (for example, 999).

We consider each possibility in turn. In the first case, the number has the form aab , where $a \neq 0$ and $b \neq a$. There are 9 possibilities for a , and for each one there are 9 possibilities for b . There are 81 such numbers in all. In the second case, the number has the form abb , where $a \neq 0$ and $b \neq a$. There are 9 possibilities for a , and for each one there are 9 possibilities for b . There are 81 such numbers in all. Finally, the third case consists of the numbers 111, 222, 333, \dots , 999. There are 9 such numbers in all. So there are $81 + 81 + 9 = 171$ rolling numbers in all.

Algebra Solutions

1. Marcus counted seven items on each lion: four legs, two ears, and one tail. Since he counted 63 items in all, Marcus saw 9 lions at the zoo.

2. Suppose there were 3 fewer girls in the class. Then there would be 26 students and half would be boys and half girls. So, in the smaller class, there would be 13 boys and 13 girls. In the real class, there are 13 boys and 16 girls.

3. We can work backward by starting with 3250, subtracting 2000 from it, and adding 40 to the answer. Since $3250 - 2000 = 1250$ and $1250 + 40 = 1290$, the number we started with is 1290.

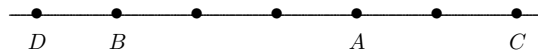
4. The given sum is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100$. To find its ones digit, we may first sum the ones digits of each term. This is $1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0 = 45$. Since the ones digit of this sum is 5, the ones digit of the original sum is also 5.

5. We need to solve $1728 = 36 \times \square$. Dividing both sides by 36 gives $\square = 48$.

6. If one cow gives 15ℓ of milk in one day, the same cow would give $8 \times 15 \ell = 120 \ell$ of milk in 8 days. Then 8 cows would give $8 \times 120 \ell = 960 \ell$ of milk in 8 days.

7. There can be only 1, 2, 3, or 4 chairs in the room. (If there were 5 chairs there would be 20 legs in the room.) If there is one chair in the room then 4 legs would belong to the chair and 13 legs would belong to stools. This is impossible. If there are two chairs in the room then 8 legs would belong to the chairs and 9 legs would belong to stools. This is possible when there are three stools in the room. If there are three chairs in the room then 12 legs would belong to the chairs and 5 legs would belong to stools. This is impossible. Finally, if there are four chairs in the room then 16 legs would belong to the chairs and only 1 leg would belong to the stools. This is impossible. Then the only possibility is that there are 2 chairs in the room.

8. We can show the information on a number line. For example, since Ania is 3 years older than Basia, we can put the letter A three spaces to the right of B . Since Ania is 2 years younger than Celina,



Celina is two years older than Ania. We can put the letter C two spaces to the right of A . Finally, since Dorota is 1 year younger than Basia we can put the letter D one space to the left of B . Now it is clear that Celina is six years older than Dorota.

9. For one student to go on each ride, the tickets cost $\$20 + \$30 + \$40 + \$50 = \$140$. Since the class tickets cost $\$3220$ all together, there were 23 students on the trip. Since each student received 4 tickets, the class bought 92 tickets in all.

10. Since three-fourths of the cost of Kristen's dinner was $\$420$, we may divide by 3 to see that one-fourth of the cost of her dinner was $\$140$. Multiplying this amount by 4, the cost of her dinner was $\$560$.

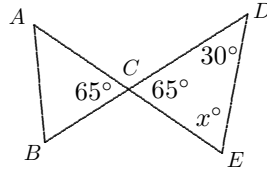
11. Suppose the shortest dwarf got b berries. The next tallest one got $b + 1$ berries, the next one got $b + 2$ berries, and so on, until the tallest one got $b + 6$ berries. Since there were 77 berries in all,

$$b + (b + 1) + (b + 2) + (b + 3) + (b + 4) + (b + 5) + (b + 6) = 77$$

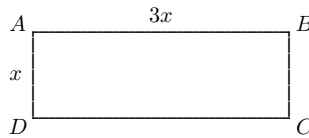
The left side above simplifies to $7b + 21$. So, $7b + 21 = 77$. Subtracting 21 from both sides, gives $7b = 56$. Dividing both sides by 7 gives $b = 8$. The shortest dwarf got 8 berries, the next tallest one got 9, and so on, until the tallest dwarf got 14 berries.

Geometry Solutions

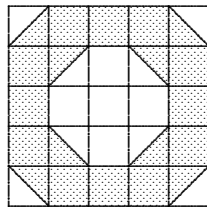
1. Let A , B , C , D , and E be the points shown in the figure below. Since $\angle ACB$ and $\angle DCE$ are vertical angles, they are equal. So, $\angle DCE = 65^\circ$. Since the sum of the angles in any triangle is 180° , $30 + 65 + x = 180$. It follows that $x = 85$.



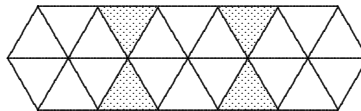
2. Let $AD = x$. Then $AB = 3x$. The perimeter of the rectangle is $3x + x + 3x + x = 8x$. But the perimeter is 120 cm. So, $8x = 120$ cm. dividing both sides by 8, $x = 15$ cm. Then $AD = 15$ cm and so $AB = 45$ cm. The area of the rectangle is $15 \text{ cm} \times 45 \text{ cm} = 675 \text{ cm}^2$.



3. There are 12 complete squares and 8 more halves of squares which are shaded. The shaded area is equivalent to $12 + 8\left(\frac{1}{2}\right) = 12 + 4 = 16$ complete squares. Since the area of each small square is 1 cm^2 , the total shaded area is 16 cm^2 .



4. Each regular hexagon consists of 6 equilateral triangles which are equal to each other. Any triangle in a hexagon is also equal to any one of the four shaded ones. The figure consists of 22 equal triangles and their



total area is 154 cm^2 . Then the area of each triangle is 7 cm^2 . Since four triangles are shaded, the total shaded area is 28 cm^2 .

5. In cm^2 , the area of the three squares combined is $(10 \times 10) + (8 \times 8) + (6 \times 6) = 100 + 64 + 36 = 200$. The total area of the figure is 200 cm^2 . The unshaded area is a triangle with height 10 cm and base $10 \text{ cm} + 8 \text{ cm} + 6 \text{ cm} = 24 \text{ cm}$. Its area is $\frac{1}{2}(24 \text{ cm})(10 \text{ cm}) = 120 \text{ cm}^2$. The shaded area is $200 \text{ cm}^2 - 120 \text{ cm}^2 = 80 \text{ cm}^2$.

