The 2014 Junior Mathematical Olympiad

Practice Problems Set 3

Logic, Pattern, and Counting Solutions

1. There are 6 small triangles as in Figure 1(a) below. There are also 2 large triangles as in Figure 1(b) below. There are a total of $6 + 2 = 8$ triangles in all.

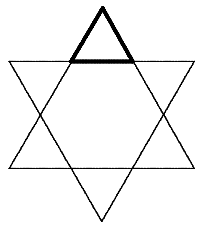

Figure 1(a)

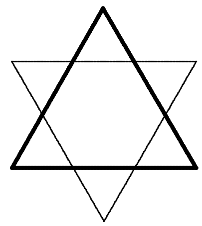

Figure 1(b)

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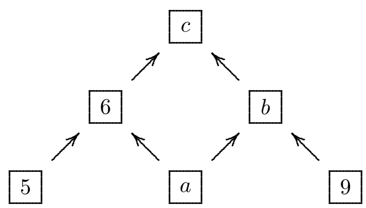
2. Let a, b, and c be the unknown numbers in the boxes, as shown in the figure below. From the pattern given, $\frac{5+a}{2} = 6$, $\frac{a+9}{2} = b$, and $\frac{6+b}{2} = c$. Multiplying each equation by 2 gives $5 + a = 12$, $a + 9 = 2b$,



and $6 + b = 2c$. Solving the first equation gives $a = 7$. Substituting this into the secoond equation and solving gives $b = 8$. Subsstituting this into the third equation and solving gives $c = 7$. The number at the top of the pyramid is 7.

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3. Suppose the three-digit number is abc. There are 4 possibilities for a: 2, 4, 6, or 8. (We cannot have $a = 0$. A number such as 082 is not a three-digit number.) For each choice for a, there are 5 possibilities for b: 0, 2, 4, 6, or 8. Once a and b are chosen, there are 5 possibilities for c: 0, 2, 4, 6, or 8. The total number of possibilities is $4 \times 5 \times 5 = 100$.

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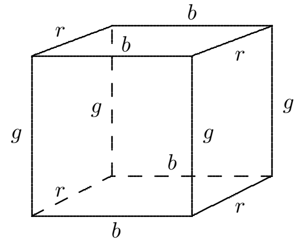
4. At each vertex, there are three sticks and each of them is a different colour. So Carlos had to use at least three colours. However, did not need more than three. For example, suppose four of the sticks were red (r),



four were blue (b), and four were green (g). They could have been arranged as shown in the figure above. So the smallest number of colours that Carlos could have used is three.

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5. The smallest possible sum is 3, which is $1+2$. The largest possible sum is 9, which is $4+5$. Furthermore, any integer between 3 and 9 may be obtained as a sum. For example, 4 is equal to $1+3$, 5 is equal to $1+4$, 6 is equal to $1+5$, 7 equal to is $2+5$, and 8 is equal to $3+5$. (Other combinations are possible. For example, 5 is also equal to $2+3$.) The distinct sums that may be obtained are 3, 4, 5, 6, 7, 8, and 9. There are seven such sums in all.

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6. If the first digit of an uphill number is 1, its second digit could be 2, 3, 4, 5, 6, 7, 8, or 9. There are 8 possibilities in all. If the first digit of an uphill number is 2, its second digit could be 3, 4, 5, 6, 7, 8, or 9. There are 7 possibilities in all. Continuing in this way,

if the first digit of an uphill number is	3,	there are	6	possibilities for its second digit	;
"	4,	"	5	"	;
"	5,	"	4	"	;
"	6,	"	3	"	;
"	7,	"	2	"	;

if the first digit of an uphill number is 8, there is 1 possibility for its second digit.

In all, there are $8+7+6+5+4+3+2+1=36$ uphill numbers with two digits.

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7. Suppose a number n has 3 digits and they sum to 5. The largest digit of n could be 5, 4, 3, or 2. (If its largest digit is 1 the sum of its digits cannot be more than 3.) We consider each possibility in turn.

If the largest digit of n is 5 thnn n has to be 500. There is only one possibility in this case.

If the largest digit of n is 4, its three digits have to be 4, 1, and 0 in some order. Then n may be 410, 401, 140, or 104. There are four possibilities in this case.

If the largest digit of n is 3, its three digits can be either 3, 2, and 0 or 3, 1, and 1. In the first case, n can be 320, 302, 230, or 203. In the second case, n can be 311, 131, or 113. In all, there are seven possibilities in this case.

If the largest digit of n is 2, the three digits of n are 2, 2, and 1. The number can be 221, 212, or 122. There are three possibilities in this case.

Combining all the cases, there are $1+4+7+3=15$ three-digit numbers whose digits sum to 5.

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8. Suppose Ariel scored more than 5 points. Then she scored 6 or more, Beatrice scored 7 or more, and Carlos scored 8 or more. They would have scored 21 or more points in this case. So, Ariel did not score more than 5 points. Suppose that Ariel scored less than 5 points. Then she scored 4 or fewer points, Carlos scored 7 or fewer, and Beatrice scored 6 or fewer. They would have scored at most 17 points. So, Ariel did not score fewer than 5 points, she scored exactly 5 points.

Then Beatrice scored at least 6 points. If she scored 6 then Carlos scored 9, and this is possible. If Beatrice scored 7 points then Carlos scored 8, and this is also possible. However, Beatrice did not score 8 or more points. Otherwise, Carlos would have scored 7 or fewer points, which is less than Ariel scored. So, the possible scores for Ariel, Beatrice, and Carlos are, respectively, 5, 6, and 9, or 5, 7, and 8.

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9. If each child receives at least 1 sweetie, there are only 2 more to hand out. So, no child will receive more than 3 sweeties. Suppose Anna receives 1 sweetie. Then Paco could receive 1 sweetie and Luis 3, or Paco could receive 2 and Luis 2, or Paco could receive 3 and Luis 1. There are three possibilities in this case. Suppose Anna receives 2 sweeties. Then Paco could receive 1 sweetie and Luis 2, or Paco could receive 2 and Luis 1. There are two possibilities in this case. If Anna receives 3 sweeties then Paco and Luis each have to receive 1. There is only one possibility in this case. The total number of possibilities is $3+2+1=6$. So, there are 6 different ways of sharing out the five sweeties.

Algebra Solutions

1. After Demario bought 3 exercise books, he had \$30 left. With another \$80, he could have bought another book. Then the cost of a book is $\$30 + \$80 = \$110$.

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2. Together, mother and daughter have $42 + 16 = 58$ mushrooms. If they both had 29 mushrooms in their basket they would have the same number. So, mother should take 13 mushrooms and give them to her daughter.

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3. There were 30 runners who finished either ahead of or behind Micalia. Since the number behind her was four times the number ahead of her, four-fifths of them finished behind her and one-fifth finished ahead of her. So, 24 runners finished behind Micalia and 6 finished ahead of her. Micalia finished 7th.

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4. The first 30 terms of the sequence consist of one 1, two 2's, three 3's, four 4's, five 5's, six 6's, seven 7's, and two 8's. Their sum is

$$(1 \times 1) + (2 \times 2) + (3 \times 3) + (4 \times 4) + (5 \times 5) + (6 \times 6) + (7 \times 7) + (2 \times 8)$$
$$= 1 + 4 + 9 + 16 + 25 + 36 + 49 + 16 = 156$$

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5. We consider each expression in turn. Using the usual order of operations,

$$2 \cdot 3 + 4 \cdot 5 = 6 + 20 = 26;$$
$$2 + 3 + 4 + 5 = 14;$$
$$(2 + 3) \cdot (4 + 5) = 5 \cdot 9 = 45;$$
$$2 \cdot (3 + 4) \cdot 5 = 2 \cdot 7 \cdot 5 = 70;$$
$$2 \cdot (3 + 4 + 5) = 2 \cdot 12 = 24$$

The least of these values is $2 + 3 + 4 + 5 = 14$.

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6. From the information given, if Joanna had baked one less cookie the total would have been divisible by 2, 3, and 4. The smallest number divisible by 2, 3, and 4 is 12. Joanna baked 13 cookies.

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7. From the information given, $\square + \bigcirc = 30$ and $\triangle + \bigcirc = 80$. Adding these equations together, $\square + \triangle + \bigcirc + \bigcirc = 110$.

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8. Starting at 12:01 am, just after midnight, and going up to 12:01 pm, just after noon, the clock will strike for 1 o'clock, 2 o'clock, and so on, up to 12 o'clock. This will amount to $1 + 2 + 3 + \cdots + 12 = 78$ strikes. The clock will also strike once for each half hour, and this will mean 12 more strikes. The clock will strike $78 + 12 = 90$ times in the morning. In the evening, from 12:01 pm until 12:01 am, the clock will strike 90 more times. So, during one 24-hour day the clock will strike $90 + 90 = 180$ times.

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9. The watermelon, all five fifths of it, weighs $\frac{4}{5}$ kg more than four fifths of the same watermelon. This means that $\frac{1}{5}$ of the watermelon weighs $\frac{4}{5}$ kg. Then $5 \times \frac{1}{5}$ of the watermelon weighs $5 \times \frac{4}{5}$ kg. So, the whole watermelon weighs 4 kg.

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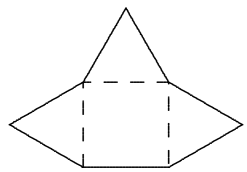
10. First, 44 days and 44 hours is equal to 45 days and 20 hours. In turn, this is equal to 6 weeks, 3 days, and 20 hours. Then Mr. Brown has lived 44 years, 44 months, 50 weeks, 3 days, and 20 hours. Also, 50 weeks, 3 days, and 20 hours is between 50 and 51 weeks. So, it is between 11 and 12 months. Then Mr. Brown has lived more than 44 years and 55 months and less than 44 years and 56 months. Since there are 12 months in each year, Mr. Brown has lived more than 48 years and 7 months and less than 48 years and 8 months. Mr. Brown is 48 years old.

Geometry Solutions

1. Each side of one of the equilateral triangles on the square has the same length. There are 7 equal sides forming the perimeter of the figure. Since the perimeter is 14 cm, each side has length 2 cm. So, each side of the square has length 2 cm. The area of the square is $4\,\text{cm}^2$.



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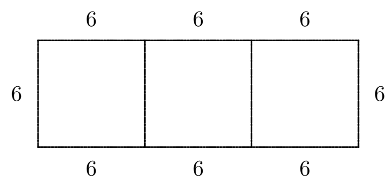
2. The perimeter of the rectangle consists of 8 equal line segments (each one is the side of an identical square). Since the total perimeter is 48 cm, each segment is 6 cm long. Then the area of one of the original squares is $6\,\text{cm} \times 6\,\text{cm} = 36\,\text{cm}^2$.



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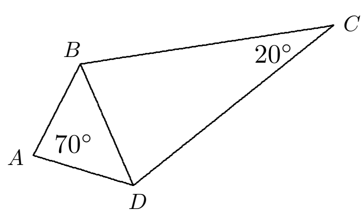
3. The triangle ABD is isosceles with $\angle ABD = \angle ADB$. The triangle CBD are isosceles with $\angle CBD = \angle CDB$. The sum of the angles in any triangle is 180°. So, in $\triangle BAD$, $70° + 2\angle ADB = 180°$. Subtracting



70° from both sides gives $2\angle ADB = 110°$. Dividing both sides by 2 gives $\angle ADB = 55°$. In $\triangle BCD$, $20° + 2\angle CDB = 180°$. It follows that $\angle CDB = 80°$. Then $\angle ADC = \angle ADB + \angle BDC = 55° + 80° = 135°$.

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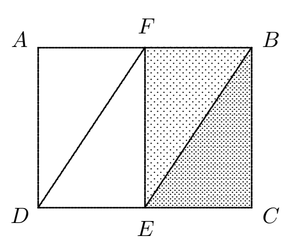
4. From E, draw a line parallel to CB which meets AB at F. The triangles BEC and BFE have the same area. Their total area is $1/4 + 1/4 = 1/2$ the area of rectangle $ABCD$. So, the area of $EBCF$ is 1/2 the



area of $ABCD$. Thus $(EC)(CB) = \frac{1}{2}(DC)(CB)$. Dividing both sides by CB gives $EC = \frac{1}{2}(DC)$.

5. The original figure may be divided into 8 equal squares. Each square is half shaded and half unshaded. Then half of the total area of the outer rectangle is shaded. Since the area of the rectangle is $3 \times 6 = 18$, the area of the shaded region is $18/2 = 9$.