The University of the West Indies, Mona Campus

Presents

The 2014 Jamaican Mathematical Olympiad

First Round

Solutions for Grades 9, 10, and 11

1. Let S be the number of sweeties Ava had in her bag. Beth received $\frac{1}{2}S = \frac{S}{2}$ sweeties from Ava. Celine received $\left(\frac{1}{3}\right)\left(\frac{S}{2}\right) = \frac{S}{6}$ sweeties from Beth. Davia received $\frac{1}{4}\left(\frac{S}{6}\right) = \frac{S}{24}$ sweeties from Celine. So, $\frac{S}{24} = 3$. Multiplying both sides by 24 gives $S = 72$. Ava had 72 sweeties in her bag.

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2. The fractions given in the problem are respectively equivalent to

$$\frac{7}{8}, \quad \frac{6}{7}, \quad \frac{5}{6}, \quad \frac{4}{5}, \quad \text{and} \quad \frac{3}{4}.$$

To compare 7/8 and 6/7, we may cross multiply and note that $49 > 48$. Dividing both sides by 56 gives $49/56 > 48/56$. Simplifying both sides gives $7/8 > 6/7$. One similarly proves that $6/7 > 5/6$, $5/6 > 4/5$ and $4/5 > 3/4$. The fraction with the greatest value is 7/8.

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3. We have $12^{mn} = (4 \cdot 3)^{mn} = 4^{mn}3^{mn} = \left(2^2\right)^{mn}3^{mn} = 2^{2mn}3^{mn} = (2^m)^{2n}(3^n)^m = P^{2n}Q^m$

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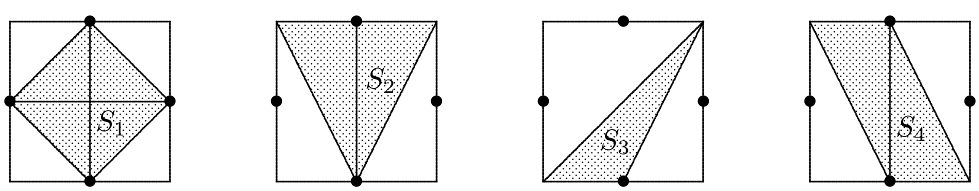
4. Let A be the area of a square. One may verify that $S_1 = \frac{1}{2}A$, $S_2 = \frac{1}{2}A$, and $S_4 = \frac{1}{2}A$. (Lines have been added to the first, second, and fourth squares below to suggest how to show this.) So,



$S_1 = S_2 = S_4$. In the third square, however, the total area below the diagonal, shaded and unshaded, is $\frac{1}{2}A$. The shaded part is only half of this area. So, $S_3 = \frac{1}{4}A$. Then $S_3 < S_1 = S_2 = S_4$.

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5. The grocer's display has one can in the top row, 3 cans in the row below it, 5 in the next one down, and so on. The grocer has 6 rows in his display because $1 + 3 + 5 + 7 + 9 + 11 = 36$.

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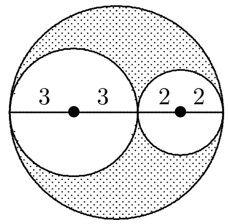
6. Let ab be a two-digit number where a and b are odd and $a \neq b$. There are 5 possibilities for a (1, 3, 5, 7, or 9). For each one, there are 4 possibilities for b (it can be any odd digit except a). There are $5 \times 4 = 20$ such numbers in all.

7. The centres of the three circles lie on the same line segment, and this may be extended to be a diameter for the largest circle. Clearly, the diameter of the outer circle is 10 and its radius is 5.



The area of the outer circle is $\pi(5)^2 = 25\pi$. The areas of the inner circles are $\pi(3)^2 = 9\pi$ and $\pi(2)^2 = 4\pi$, respectively. Their combined area is $9\pi + 4\pi = 13\pi$. The area of the shaded region is the area of the outer circle minus the combined area of the inner circles. This is $25\pi - 13\pi = 12\pi$.

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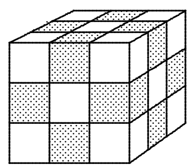
8. When the 27 dice are glued together, 8 of them will be in a corner of the cube. Each one will have three faces exposed. The smallest numbers that could be showing on them are 1, 2, and 3.



This gives a minimum sum of 6 for each corner die. There will also be 12 dice on the edges of the cube that lie between corner pieces. (These are the shaded dice in the figure above.) Each one has two sides exposed. The smallest numbers that could be showing on them are 1 and 2. This gives a minimum sum of 3 for each die on an edge. There will also be six dice, one on each face of the cube, which have one side exposed. The smallest possible number on the exposed side is 1. The smallest possible sum of the numbers on the surface of the $3 \times 3 \times 3$ cube is $8(6) + 12(3) + 6 = 90$.

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9. Let x be the age of the injured player, and let a_1, a_2, a_3, $\ldots$, a_{10} be the ages of the other ten Strikers. Since the average age of the whole team is 22,

$$\frac{a_1 + a_2 + a_3 + \cdots + a_{10} + x}{11} = 22.$$

Multiplying both sides by 11 gives

$$a_1 + a_2 + a_3 + \cdots + a_{10} + x = 242. \quad (1)$$

After the injured player left, the average age of the 10 remaining players was 21. So,

$$\frac{a_1 + a_2 + a_3 + \cdots + a_{10}}{10} = 21.$$

Multiplying both sides by 10 gives

$$a_1 + a_2 + a_3 + \cdots + a_{10} = 210. \quad (2)$$

Subtracting Equation (2) from (1) gives $x = 32$. The injured player was 32 years old.

10. Let x be the number of \$10 coins and y the number of \$20 coins in the box. Then $x = 3y$. After removing 8 coins of each type from the box, we would have $x - 8 = 5(y - 8) = 5y - 40$. Adding 8 throughout gives $x = 5y - 32$. Since $x = x$ we have $3y = 5y - 32$. It follows that $2y = 32$ and so $y = 16$. Also, $x = 3y = 3(16) = 48$. There are 48 \$10 coins and 16 \$20 coins in the box. Their value, in dollars, is $48(10) + 16(20) = 480 + 320 = 800$.

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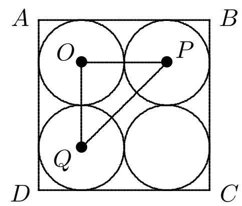
11. Let $ABCD$ be the vertices of the square, and let O be the centre of a third circle as shown below. Since the points O and Q are equally distant from the side AD, OQ is parallel to AD.



Similarly, OP is parallel to AB. Then $\triangle POQ$ has a right angle at O. It is clear that $2a$, the length of a side of the square, is equal to twice the diameter of one of the circles. It follows that each circle has diameter a and radius $a/2$. Then $OP = a/2 + a/2 = a$ and $OQ = a/2 + a/2 = a$. By the Pythagorean theorem, $(PQ)^2 = (PO)^2 + (OQ)^2 = a^2 + a^2 = 2a^2$. Thus $PQ = \sqrt{2}\,a$.

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12. First,

$$\begin{aligned} p &= (1)(2)(4) + (2\cdot 1)(2\cdot 2)(2\cdot 4) + (3\cdot 1)(3\cdot 2)(3\cdot 4) + \cdots + (2014\cdot 1)(2014\cdot 2)(2014\cdot 4) \\ &= (1)(2)(4) + 2^3(1)(2)(4) + 3^3(1)(2)(4) + \cdots + 2014^3(1)(2)(4) = (1^3 + 2^3 + \cdots + 2014^3)(8) \end{aligned}$$

In the same way, $q = (1^3 + 2^3 + 3^3 + \cdots + 2014^3)(27)$. Then

$$\frac{p}{q} = \frac{(1^3 + 2^3 + 3^3 + \cdots + 2014^3)(8)}{(1^3 + 2^3 + 3^3 + \cdots + 2014^3)(27)} = \frac{8}{27}.$$

Thus $\sqrt[3]{p/q} = \sqrt[3]{8/27} = 2/3$.