THE UNIVERSITY OF THE WEST INDIES, MONA

The 2015 Jamaican Mathematical Olympiad

Solutions for Grades 9, 10, and 11

1. The numbers are listed from least to greatest. So, their median is the number in the middle (i.e., the fifth number on the list). Then $n + 6 = 10$, and hence $n = 4$. The numbers on the list are 4, 7, 8, 9, 10, 12, 14, 16, and 19. Their average is

$$\frac{4+7+8+9+10+12+14+16+19}{9} = \frac{99}{9} = 11.$$

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2. We have

$$\frac{10^{2013}+10^{2015}}{10^{2014}+10^{2014}} = \frac{10^{2013}(1+10^2)}{10^{2013}(10+10)} = \frac{1+100}{10+10} = \frac{101}{20} = 5.05$$

This ratio is closer to 5 than to any other number given as a choice.

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3. We have $E = 2+4+6+\cdots+4028$ and $D = 1+3+5+\cdots+4027$. Then $E - D = (2+4+6+\cdots+4028) - (1+3+5+\cdots+4027)$. Rearranging terms gives

$$\begin{aligned} E - D &= (2-1)+(4-3)+(6-5)+\cdots+(4028-4027) \\ &= 1+1+1+\cdots+1 \quad \text{(2014 times)} \\ &= 2014. \end{aligned}$$

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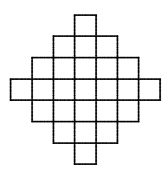
4. The original polygon has 28 equal sides and a perimeter of 56. So, each side has length 2. The interior of the polygon contains 25 equal squares (see the figure below). Each square has a side length of 2 and an area of 4. The area of the original polygon is $25 \times 4 = 100$.



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5. Let x be the original number. Cindy subtracted 9 from x and divided by 3, obtaining 43. So, $(x-9)/3 = 43$. Multiplying both sides by 3 gives $x - 9 = 129$. Adding 9 to both sides gives $x = 138$. The original number was 138. Cindy should have subtracted 3, obtaining 135, and then divided by 9. She should have obtained 15.

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6. The two-digit square numbers are 16, 25, 36, 49, 64, and 81. The numbers 16, 36, 64, and 81 are the first two digits of the supersquares 164, 364, 649, and 816. The numbers 25 and 49 are

not the first two digits of any supersquare. So there are 4 supersquares in all, and their sum is $164 + 364 + 649 + 816 = 1993$.

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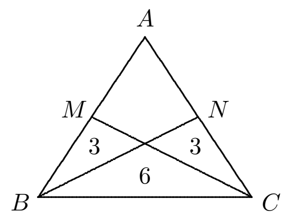
7. We may re-orient the diagram so that AC is considered the base of $\triangle ACB$. Then the triangles ANB and BNC have equal bases and equal (perpendicular) heights. Thus they have equal areas.



In particular, the area of $\triangle ANB$ is 9. Thus the area of the quadrilateral inside $\triangle ABC$ is 6.

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8. Taking advantage of the sale, Adam bought 7 chocolates for the price of 6, which is \$24. At the same rate, he bought 14 chocolates for \$48, 21 chocolates for \$72, and so on. Eventually, he bought 98 chocolates for \$336. The next 2 chocolates cost \$8. So, he bought 100 chocolates for \$344. He had \$56 left after that.

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9. Let x be the original number. When the decimal point in the representation of x is moved four places to the right, the new number has the value $10,000x$. Since this is four times the reciprocal of x, $10,000x = 4/x$. Multiplying both sides by x gives $10,000x^2 = 4$. Taking the positive square root of both sides gives $100x = 2$. Dividing throughout by 100 gives $x = 0.02$.

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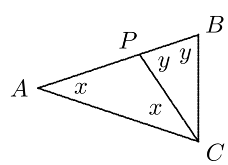
10. Let $x = \angle BAC$ and $y = \angle ABC$. Since $\triangle APC$ is isosceles with $AP = PC$, $\angle PCA = x$ as well. Since $\triangle PCB$ is isosceles with $PC = CB$, $\angle CPB = y$ as well. The sum of the angles in $\triangle APC$ is $180°$. Thus $\angle APC = 180° - 2x$. Also, $\angle APB$ is a straight angle. Thus $(180° - 2x) + y = 180°$.



Solving for y gives $y = 2x$. Thus $\angle ABC = 2x$. Since $\triangle ABC$ is isosceles with $AB = AC$, $\angle ACB = 2x$ as well. Since the sum of the angles in $\triangle ABC$ is $180°$, $x + 2x + 2x = 180°$. That is, $5x = 180°$. Dividing both sides by 5 gives $x = 36°$.

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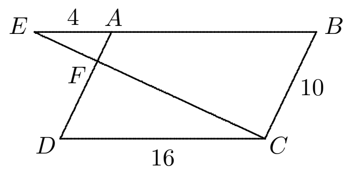
11. In sixty minutes the minute hand of a clock sweeps out one full circle, which is $360°$. In one minute it sweeps out one-sixtieth of this, which is $6°$. In sixty minutes, the hour hand sweeps out one-twelfth of a circle, which is $30°$. In one minute it sweeps out one-sixtieth of this, which is $0.5°$. At 9:00, the minute hand is at $0°$ and the hour hand is at $270°$. (We may liken a clock face to a compass, with 12:00 representing due north.) Twenty minutes later, at 9:20, the minute hand has advanced by $120°$ and the hour hand has advanced by $10°$. Their new positions are $120°$ and $280°$, respectively. The difference between these angles is $280° - 120° = 160°$.

12. Triangles EAF and CDF are similar. This is because $\angle AEF = \angle AEC = \angle ECD = \angle DCF$ and $\angle EAF = \angle EAD = \angle ADC = \angle FDC$. (In both cases these are alternate angles between the



parallel lines EB and DC). Also, $\angle EFA = \angle CFD$ by vertical angles. Since corresponding sides of similar triangles are in the same proportion,

$$\frac{AF}{FD} = \frac{EA}{CD} = \frac{4}{16} = \frac{1}{4}$$

Cross multiplying gives $FD = 4(AF)$. Since opposite sides of a parallelogram are equal, $AD = 10$. This means that $AF + FD = 10$ and hence $AF + 4(AF) = 10$. Thus $5(AF) = 10$ and hence $AF = 2$.

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13. If x and y are positive integers with a highest common factor (HCF) of 5, both of them are positive multiples of 5. It follows that $x, y \in \{5, 10, 15, \cdots, 55\}$. The ordered pairs (x, y) such that $x + y = 60$ are

$$(5,55),\ (10,50),\ (15,45),\ (20,40),\ (25,35),\ (30,30),\ (35,25),\ (40,20),\ (45,15),\ (50,10),\ (55,5)$$

The HCF of x and y is 5 in exactly four of these pairs. These are $(5,55)$, $(25,35)$, $(35,25)$, and $(55,5)$.

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14. Let the colour for the upper left corner be denoted by a. The square next to it (in the top row) must have a different colour, call it b. The square below it must also have a colour different

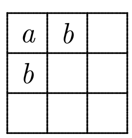

Figure 14(a)

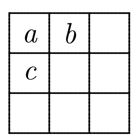

Figure 14(b)

from a. It may be either b or c (see the figures above). In either case, once these three squares are coloured there is only one way to colour the remaining squares. (Try it.) In the first case, as in Figure 14(a), there are 3 choices for a and two choices for b. There are $3 \times 2 = 6$ different colourings in this case. In the second case, as in Figure 14(b), there are again 3 choices for a and two choices for b. There are $3 \times 2 = 6$ different colourings in this case. Then there are $6 + 6 = 12$ different colourings in all.

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15. Since $xy = 24$ and $xz = 48$, we may divide equals into equals to see that

$$\frac{xy}{xz} = \frac{24}{48} \qquad \text{and hence} \qquad \frac{y}{x} = \frac{1}{2}$$

Cross multiplying gives $z = 2y$. Since $yz = 72$ we may substitute for z to see that $2y^2 = 72$. Then $y^2 = 36$ and hence $y = 6$. (It is given that y is a positive real number.) It follows that $x = 4$ and $z = 12$. Thus $x + y + z = 4 + 6 + 12 = 22$.

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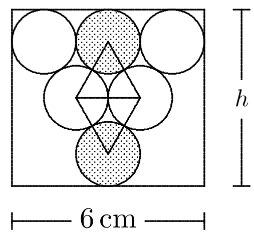
16. In the top row, the diameters of the three circles span 6 cm. So, the diameter of each circle is 2 cm and the radius of each one is 1 cm. The centres of the top shaded circle and the two unshaded



circles below it are vertices of an equilateral triangle. Each side of this triangle is 2 cm long. The altitude of this triangle is $\sqrt{3}$ cm. (This may be deduced from the Pythagorean theorem.) In a similar way, the centres of the bottom shaded circle and the two unshaded circles above it form an equilateral triangle with altitude $\sqrt{3}$ cm. It follows that $h = 1 + \sqrt{3} + \sqrt{3} + 1 = 2 + 2\sqrt{3}$. From the top of the rectangle to the bottom of the upper shaded circle is 2 cm. From the bottom of the rectangle to the top of the lower shaded circle is also 2 cm. Then the shortest distance between the shaded circles is $(2 + 2\sqrt{3}) - 4 = 2\sqrt{3} - 2$.

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17. Dividing the equation $1001C - 2002A = 4004$ throughout by 1001 gives $C - 2A = 4$. Adding $2A$ to both sides gives $C = 2A + 4$. Dividing the equation $1001B + 3003A = 5005$ throughout by 1001 gives $B + 3A = 5$. Subtracting $3A$ from both sides gives $B = -3A + 5$. Let m be the average of A, B, and C. Then

$$m = \frac{A + B + C}{3} = \frac{A + (-3A + 5) + (2A + 4)}{3} = \frac{9}{3} = 3.$$

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18. On the first step, 10 tiles are removed (numbered 1, 4, 9, ..., 100) and so 90 tiles remain. On the second step, 9 tiles are removed (numbered 1, 4, 9, ..., 81) and so 81 tiles remain. On the next steps 9, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, and 1 tiles are removed. At this stage, there is one tile remaining. The operation must be preformed 18 times to leave 1 tile remaining.

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19. The sum $1 + 2 + 3 + \cdots + n$, which is the sum of the first n positive integers, is equal to $n(n + 1)/2 = (n^2 + n)/2$. This will evenly divide $6n$ when $6n$ is a multiple of $(n^2 + n)/2$. That is, $1 + 2 + 3 + \cdots + n$ evenly divides $6n$ when $6n = k(n^2 + n)/2$ for some positive integer k. This holds if and only if $12n = k(n^2 + n)$ for some positive integer k. In turn, dividing both sides by n, this holds if and only if $12 = k(n+1)$ for some k. Since k is a positive divisor of 12, $k \in \{1, 2, 3, 4, 6, 12\}$. If $k = 1$ then $n = 11$; if $k = 2$ then $n = 5$; if $k = 3$ then $n = 3$; if $k = 4$ then $n = 2$, and if $k = 6$ then $n = 1$. The choice $k = 12$ is not possible because $n = -1$ in this case. Then there are five numbers n such that $1 + 2 + \cdots + n$ evenly divides $6n$. These are $n = 1$, 2, 3, 5, and 11.

20. Let $n = ab23$ be a four-digit number that is divisible by 3. If $a = 1$ then n has the form $1b23$. This is divisible by 3 when $b = 0$, 3, 6, or 9. There are four such numbers in this case. If $a = 2$ then n has the form $2b23$. This is divisible by 3 when $b = 2$, 5, or 8. There are three such numbers in this case. If $a = 3$ then n has the form $3b23$. This is divisible by 3 when $b = 1$, 4, or 7. There are three such numbers in this case. Continuing in this way, if $a = 4$ there are four possible numbers, if $a = 5$ there are three possible numbers, and so on. There are $4+3+3+4+3+3+4+3+3 = 30$ possible numbers in all.

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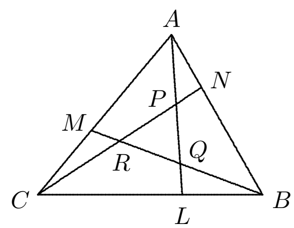
21. Let $x = AL + BM + CN$ and let P, Q, and R be the points of intersection shown below. The



sum of the perimeters the three quadrilaterals is 25. (All dimensions in this solution are centimetres.) Thus

$$(AN + MR + RP + PA) + (CL + LQ + QR + RC) + (BN + NP + PQ + QB) = 25$$

Rearranging terms gives

$$AM + CL + BN + (AP + PQ + QL) + (BQ + QR + RM) + (CR + RP + PN) = 25.$$

Thus $AM + CL + BN + AL + BM + CN = 25$. That is, $AM + CL + BN + x = 25$. Also, the sum of the perimeters of the four interior triangles is 20. Thus

$$(MC + CR + RM) + (LB + BQ + QL) + (NA + AP + PN) + (PR + RQ + QP) = 20.$$

Rearranging terms gives

$$MC + LB + NA + (AP + PQ + QL) + (BQ + QR + RM) + (CR + RP + PN) = 20.$$

Thus $MC + LB + NA + AL + BM + CN = 20$. That is, $MC + LB + NA + x = 20$. We therefore have $AM + CL + BN + x = 25$ and $MC + LB + NA + x = 20$. Adding these equations gives $(AM + MC) + (CL + LB) + (BN + NA) + 2x = 45$. That is, $AC + CB + BA + 2x = 45$. But $AC + CB + BA$ is the perimeter of $\triangle ABC$, and this is 19. Thus $19 + 2x = 45$. Subtracting 19 from both sides gives $2x = 26$. Dividing both sides by 2 gives $x = 13$. Thus $AL + BM + CN = 13$.

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22. The original expression may be written as the four equations

$$a + 1 = b + 2, \qquad a + 1 = c + 3, \qquad a + 1 = d + 4, \quad \text{and} \quad a + 1 = a + b + c + d + 5.$$

From the first three, $b = a - 1$, $c = a - 2$, and $d = a - 3$, respectively. Substituting these expressions into the fourth equation gives $a + 1 = a + (a - 1) + (a - 2) + (a - 3) + 5$. Thus

$a+1=4a-1$. Subtracting a from both sides gives $1=3a-1$. Adding 1 to both sides gives $2=3a$ and hence $a=2/3$. Then $b=-1/3$, $c=-4/3$, and $d=-7/3$. Thus

$$a+b+c+d=\frac{2}{3}-\frac{1}{3}-\frac{4}{3}-\frac{7}{3}=-\frac{10}{3}$$

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23. Let t be the time, measured in hours, after 8:00 am that Mr. Bird's work begins. Let d be the distance measured in kilometres from his house to his workplace. When Mr. Bird drives 40 k/h, he is 3 minutes late. Since rate times time equals distance, and 3 minutes is one-twentieth of an hour,

$$40\left(t+\frac{1}{20}\right)=d \text{ and so } 40t+2=d.$$

When Mr. Bird drives 60 k/h, he is 3 minutes early. Thus

$$60\left(t-\frac{1}{20}\right)=d \text{ and so } 60t-3=d.$$

Eliminating d from these equations gives $40t+2=60t-3$. Subtracting $40t$ and adding 3 to both sides gives $5=20t$. Thus $t=1/4$ (measured in hours), and his work begins at 8:15 am. We may solve either of the first two equations to see that $d=12$ km. Then to arrive on time Mr. Bird must have $r(1/4)=12$. Multiplying both sides by 4 gives $r=48$. Mr. Bird should drive 48 k/h to arrive at work on time.

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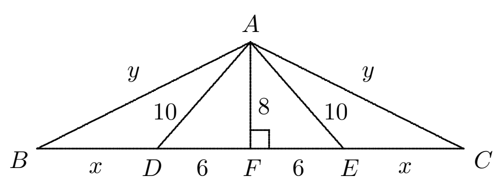
24. Let $x=BD=EC$ and let AF be an altitude for $\triangle ADE$. Since $\triangle ADE$ is isosceles, $DF=FE=6$. By the Pythagorean theorem, $(AF)^2+6^2=10^2$. Solving for AF gives $AF=8$. Let $y=AB$. By the Pythagorean theorem again,

$$y^2=(x+6)^2+8^2=(x^2+12x+36)+64=x^2+12x+100.$$

In the same way, $(AC)^2=x^2+12x+100$. Thus $AC=y$ as well. The perimeter of $\triangle ABC$ is twice the perimeter of $\triangle ADE$. This means that $2x+2y+12=2(32)=64$. Dividing by 2 throughout gives $x+y+6=32$. Subtracting $x+6$ from both sides gives $y=26-x$. Finally, squaring both sides gives $y^2=676-52x+x^2=x^2-52x+676$. Thus we have $y^2=x^2+12x+100$ and $y^2=x^2-52x+676$. Eliminating y^2 gives $x^2+12x+100=x^2-52x+676$. Subtracting x^2 from both sides gives $12x+100=-52x+676$. Adding $52x$ to both sides gives $64x+100=676$. Subtracting 100 from both sides gives $64x=576$. Finally, dividing both sides by 64 gives $x=9$. Thus $BD=9$.



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25. Since n is a multiple of 36, it is a multiple of both 4 and 9. Also, the digits of n must include at least one 4 and one 9. In order for n to be divisible by 9, the sum of its digits must be divisible by 9. This means that the number of 4's among its digits must be a multiple of 9. So, the digits of n must include one 9 and nine 4's. Its last two digits must be 44, 49, 94, or 99. Since n is a multiple of 4, the only possibility is 44. The smallest number that meets the stated criteria is $n=4,444,444,944$. The last four digits of n (written as a four-digit number) are 4944. [Note: a number $abcd\cdots efg$ is divisible by 4 if and only if the two-digit number fg is divisible by 4.]