The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problems Set 1

1. On the first day, Nikeshia ate 1 sweetie. On the second day she ate 2 sweeties. On the third day she ate 3 sweeties, and so on. During the first week, she ate $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ sweeties.

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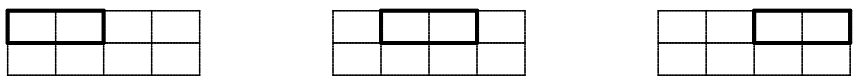
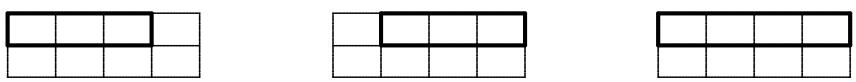
2. In the top row, there are 4 small rectangles as shown here:

There are 3 more rectangles consisting of two small ones side-by-side:



There are 2 more rectangles consisting of three small ones side-by-side, and 1 more which is the entire top row of the diagram.



Then there are $4 + 3 + 2 + 1 = 10$ rectangles in the top row. In the same way, there are 10 rectangles in the bottom row. Finally, there are 10 more rectangles which span both rows. Four of there are shown here:

A total of 30 rectangles may be found in the diagram.

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3. After Anna gave two-thirds of the pizza to her sister, she had one-third of it left. After she gave half of that to her mother, the other half was the portion that she ate. Then Anna ate one-half of one-third of the pizza. This is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ of the total pizza.

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4. The sixth symbol in the sequence is the spade suit (♠). Since the pattern repeats every six symbols, the 12th, 18th, 24th, and so on, symbols are all spade suits. This will continue through the 96th symbol, which is a spade suit again. The 97th symbol is the heart suit (♡), the 98th symbol is the club suit (♣), and the 99th symbol is the heart suit. The 100th symbol is the diamond suit (♢) and the 101st symbol is the heart suit. The answer is ♢ ♡.

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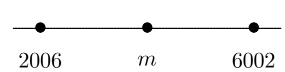
5. If m is half way between 2006 and 6002 (as shown in the figure on the right) then $m - 2006 = 6002 - m$. Thus $2m = 8008$ and so $m = 4004$.



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6. The shaded region consists of 5 small squares completely shaded, four small shaded triangles as in Figure 6(b), and one larger shaded triangle as in Figure 6(c). (See the figure on the next page.) The area of each small shaded square is $1\,\text{cm}^2$. Each small shaded triangle has area $1/2\,\text{cm}^2$. The larger shaded triangle

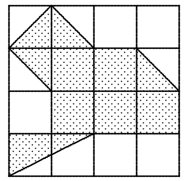

Fig. 6(a)

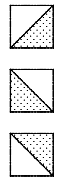

Fig. 6(b)

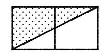

Fig. 6(c)

is half of the rectangle that contains it. The area of the rectangle is $2\,\text{cm}^2$, and the area of the shaded triangle is $1\,\text{cm}^2$. The total shaded area, in square centimetres, is $5(1) + 4(1/2) + 1 = 5 + 2 + 1 = 8$.

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7. There are 5 odd digits that could be the first number in the combination. (They are 1, 3, 5, 7, and 9.) For each one, there are 4 others that could be the second number in the combination. For each pair chosen to that point, there are 3 other digits that could be the last number in the combination. There are $5 \times 4 \times 3 = 60$ possible combinations in all. To check this, note that there are 12 combinations whose first number is 1:

1-3-5	1-5-3	1-7-3	1-9-3
1-3-7	1-5-7	1-7-5	1-9-5
1-3-9	1-5-9	1-7-9	1-9-7

Similarly, there are 12 combinations whose first digit is 3, 12 whose first digit is 5, 12 whose first digit is 7, and 12 whose first digit is 9. This gives 60 combinations in all.

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8. The multiples of three from 1 to 100 are 3, 6, 9, 12, ..., 99. These are the numbers 3×1, 3×2, 3×3, 3×4, ..., 3×33. So when you clap your hands on the multiples of 3, you clap 33 times in all. The numbers ending in 3 are

3, 13, 23, 33, 43, 53, 63, 73, 83, and 93

The numbers 3, 33, 63, and 93 are multiples of 3 and have already been counted. The other numbers, 13, 23, 43, 53, 73, and 83, are not multiples of 3. These are 6 more numbers that you clap your hands on. So there are $33 + 6 = 39$ numbers that you clap your hands on.

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9. The smallest three-digit number is 100. However, not all of its digits are different. The next smallest number is 101, but not all of its digits are different either. The next smallest number is 102, and all of its digits are different. The largest three-digit number is 999, but not all of its digits are different. None of the next largest numbers, 998, 997, 996, ..., down to 988, have all different digits either. The largest number with all different digits is 987. Then Shania subtracted 102 from 987 and obtained 885.

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10. We know that 6 chicks eat 8 cups of grain in 3 days. During the same time, half as many chicks would eat half as much grain. So, 3 chicks would eat 4 cups of grain in 3 days. These chicks would eat three times as much grain during three times as many days. So 3 chicks would eat 12 cups of grain in 9 days. (Note: One can determine that 1 chick eats 4/9 cups of grain in 1 day.)

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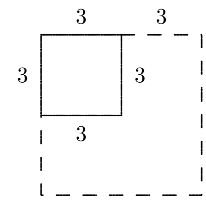
11. Suppose the towns are called A, B, C, D, E, F, and G. Since A is connected to each of the other towns, there are 6 roads from A to another town. (There is one from A to B, one from A to C, and so on.) There are 5 more roads from B to another town. (These do not include the one from B to A, which has already been counted). There are four more roads from C to another town, three from D to another town, two from E to another town, and one from F to G. There are a total of $6 + 5 + 4 + 3 + 2 + 1 = 21$ roads in Olympiette.

12. Akeem's uncle needs 3 minutes to walk along each side of the smaller plaza. The sides of the larger plaza are twice as long as the sides of the smaller one. His uncle will need 6 minutes to walk along each of side of the larger plaza. He will need $6+6+6+6=24$ minutes to walk around the larger plaza.



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13. Since $b=6$ the number to be found has the form $a6cd$. Since this number is a multiple of both 4 and 5, it is a multiple of 20. Thus it has the form $a600$, $a620$, $a640$, $a660$, or $a680$. We consider each possibility in turn. Since a is twice as much as c, the number in the first case would be 0600. However, this is not a four-digit number and must be discarded. The number in the second case would be 4620. This is also a 3-multiple and so it satisfies all stated criteria. The number in the third case would be 8640. This is also 3-multiple and it also satisfies all stated criteria. The numbers in the last two cases are impossible because no single digit a can be twice as much as 6 or 8. There are two numbers that satisfy all stated criteria: 4620 and 8640.

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14. For a whole number to be divisible by both 7 and 9, it must be a multiple of both 7 and 9. But a multiple of both 7 and 9 is a multiple of 63. The multiples of 63 between 600 and 800 are 630, 693, and 756. The only odd multiple is 693. The sum of its digits is $6+9+3=18$.

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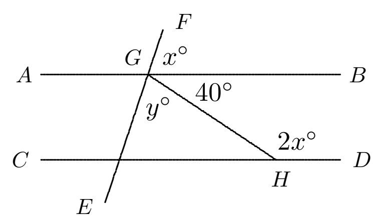
15. The lines AB and CD are parallel and the line GH crosses them both. Then the angles BGH and GHD are supplementary. That is, $40^\circ + 2x^\circ = 180^\circ$. Then $2x^\circ = 180^\circ - 40^\circ = 140^\circ$ and hence $x = 70^\circ$. Since FGE is a straight line, the angles FGB, BGH, and HGE sum to 180°. That is, $x^\circ + 40^\circ + y^\circ = 180^\circ$. Then $70^\circ + 40^\circ + y^\circ = 180^\circ$ and so $110^\circ + y^\circ = 180^\circ$. Thus $y^\circ = 180^\circ - 110^\circ = 70^\circ$.



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16. Let the ages of Abigail, Brianna, and Christina be denoted by A, B, and C, respectively. From the information given, $A+B=25$, $A+C=20$, and $B+C=31$. From the first two equations, $A=25-B$ and $A=20-C$. Then $25-B=20-C$ and so $B-C=5$. Adding this to the third equation gives $2B=36$ and so $B=18$. Substituting this into the original equations gives $A=7$ and $C=13$. Then Abigail is 7, Brianna is 18, and Christina is 13. In particular, Brianna is the oldest and she is 18.

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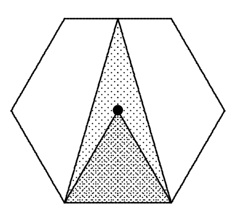
17. Consider a triangle with the same base as the given triangle and with one vertex being the centre of the circle. (For example, consider the shorter, darker triangle in the figure below.) The area of the shorter triangle is 1/6 the area of the hexagon. The taller triangle



has twice as much area because it has the same base as the shorter one and twice its height. Then the proportion of the area of the hexagon which is contained in the taller triangle is

$$2\left(\frac{1}{6}\right) = \frac{2}{6} = \frac{1}{3}$$

18. From the information given, $4^{xy} = (4^x)^y = 9^y = 256$. On the other hand, we also have $4^4 = 4\times4\times4\times4 = 256$. Then $4^{xy} = 4^4$ and it follows that $xy = 4$.

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19. Using the usual multiplication algorithm,

$$111,111,111 \times 111,111,111 = 12,345,678,987,654,321$$

The sum of the digits in this product is $1 + 2 + 3 + \cdots + 8 + 9 + 8 + 7 + \cdots + 1 = 81$.

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20. From the information given, $M = 0.30Q$, $Q = 0.20P$, and $N = 0.50P$. Then $M = 0.3Q = 0.3(0.2P) = 0.06P$. Since $N = 0.5P$,

$$\frac{M}{N} = \frac{0.06P}{0.5P} = \frac{0.06}{0.5}$$

Multiplying top and bottom by 100 gives $\frac{M}{N} = \frac{0.06}{0.5}\left(\frac{100}{100}\right) = \frac{6}{50} = \frac{3}{25}$.

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21. Half of the tulips are yellow and half are not. So, 30 tulips are yellow and 30 are not. Among the 30 tulips not yellow, one-third are red. So, 10 tulips are red. Thus $30 + 10 = 40$ tulips are either yellow or red. This leaves 20 tulips of neither colour. Among them, one-fourth are pink. So, 5 tulips are pink. Thus $30+10+5 = 45$ tulips are yellow, red, or pink. The remaining 15 tulips are white. The fraction of the tulips that are white is $15/60 = 1/4$.

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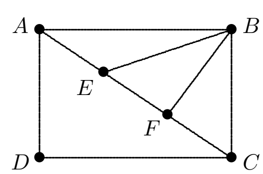
22. From the information given, the rectangle $ABCD$ has area 15. It follows that the triangle ABC has area $15/2$. Furthermore, the triangles BAE, BEF, and BFC all have



equal bases and equal heights. Thus they have equal areas. Then the area of BEF is $1/3$ the area of ABC, and this is

$$\frac{1}{3} \times \frac{15}{2} = \frac{15}{6} = \frac{5}{2}.$$

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23. From the first two equations, $b = \frac{2}{a}$ and $b = \frac{3}{c}$. Then $\frac{2}{a} = \frac{3}{c}$ and it follows that $c = \frac{3a}{2}$. From the next two equations, $d = \frac{4}{c}$ and $d = \frac{5}{e}$. Then $\frac{4}{c} = \frac{5}{e}$ and it follows that $c = \frac{4e}{5}$. Since $c = c$ we have $\frac{4e}{5} = \frac{3a}{2}$. Then $8e = 15a$ and so $\frac{e}{a} = \frac{15}{8}$.

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24. Suppose first that a, b, c, and d are any positive numbers. If $a/b > c/d$ then, multiplying both sides by bd (which is positive), $ad > bc$. On the other hand if $ad > bc$ then, multiplying both sides by $1/bd$, we have $a/b > c/d$. Thus we may say that

$$\frac{a}{b} > \frac{c}{d} \text{ if and only if } ad > bc$$

We now show that $a/(b-1)$ is larger than any of the other four quantities. First, $a/(b-1) > a/(b+1)$. This is because $b-1 < b+1$ and when a is divided by the smaller quantity its quotient is larger. Second, $a/(b-1) > 2a/(2b+1)$. To see this, cross multiply to get $a(2b+1) > (b-1)(2a)$ and hence $2ab+a > 2ab-2a$. Subtracting $2ab$ from both sides gives $a > -2a$. Since a is positive, this is true and validates the inequality we started with. Third, $a/(b-1) > (2a)/(2b-1)$. To see this, cross multiply to get $a(2b-1) > (b-1)(2a)$ and hence $2ab - a > 2ab - 2a$. Subtracting $2ab$ from both sides gives $-a > -2a$ and hence $a < 2a$. Since a is positive, this is true and validates the inequality we started with. Finally, $a/(b-1) > 3a/(3b+1)$. To see this, cross multiply to get $a(3b+1) > (b-1)(3a)$ and hence $3ab + a > 3ab - 3a$. Subtracting $3ab$ from both sides, $a > -3a$. Since a is positive, this is true and validates the inequality we started with. In summary, the largest of the five quantities is $a/(b-1)$.

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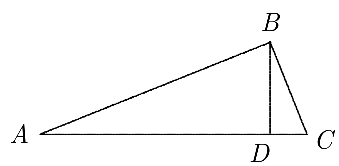
25. The triangles ADB and ABC are similar. This is because both are right triangles with a common angle at A. Since the sum of the angles in $\triangle ADB$ and in $\triangle ABC$ are the same (both sums are $180°$),

$$90° + \angle A + \angle ABD = 90° + \angle A + \angle ACB.$$

It follows that $\angle ABD = \angle ACB$ as well. Since corresponding sides in similar triangles are in the same proportion, $AD/AB = AB/AC$. Cross-multiplying gives $(AD)(AC) = (AB)^2$. Substituting 64 for AD and 100 for AC gives $(AB)^2 = 6400$. Thus $AB = 80$.



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26. The two-digit perfect squares are 16, 25, 36, 49, 64, and 81. So suppose every pair of consecutive digits of a number is a perfect square. If the digit 1 is present the next digit must be 6, when 2 is present the next digit must be 5, and so on. Note that whenever 5, 7, or 9 occurs there cannot be a next digit. Among other things, this means that the first digit of the number cannot be 5, 7, or 9. We consider each of the possibilities 1, 2, 3, 4, 6, and 8 in turn. If the first digit of the number is 1 it has to be 1649. If its first digit is 2 it has to be 25. If its first digit is 3 it has to be 3649. If its first digit is 4 it has to be 49. If its first digit is 6 it has to be 649. And if its first digit is 8 it has to be 81649. The longest such number is 81649 and it has five digits.

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27. Suppose there are d doctors and ℓ lawyers in the group. Let $a_1, a_2, \ldots, a_d$ be the ages of the doctors and $b_1, b_2, \ldots, b_\ell$ the ages of the lawyers. Since the average age of the doctors is 35,

$$\frac{a_1 + a_2 + \cdots + a_d}{d} = 35.$$

Multiplying both sides by d gives $a_1 + a_2 + \cdots + a_d = 35d$. Since the average age of the lawyers is 50,

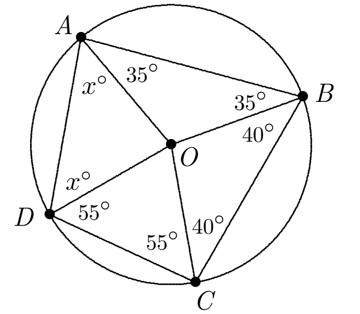
$$\frac{b_1 + b_2 + \cdots + b_\ell}{\ell} = 50.$$

Multiplying both sides by ℓ gives $b_1 + b_2 + \cdots + b_\ell = 50\ell$. Since the average age of the whole group is 40,

$$\frac{(a_1 + a_2 + \cdots + a_d) + (b_1 + b_2 + \cdots + b_\ell)}{d + \ell} = 40.$$

Multiplying both sides by $d + \ell$ gives $(a_1 + a_2 + \cdots + a_d) + (b_1 + b_2 + \cdots + b_\ell) = 40(d + \ell)$. The left side is equal to $35d + 50\ell$. The right side is equal to $40d + 40\ell$. So, $35d + 50\ell = 40d + 40\ell$. Subtracting $35d + 40\ell$ from both sides gives $10\ell = 5d$. Then $d/\ell = 10/5 = 2/1$. Thus $d : \ell = 2 : 1$.

28. Let $\angle ODA = x°$. Note that $OA = OB = OC = OD$ because OA, OB, OC, and OD are all radial segments. Then $\triangle OAB$, $\triangle OBC$, $\triangle OCD$, and $\triangle ODA$ are isosceles. Thus $\angle OAB = \angle OBA = 35°$, $\angle OBC = \angle OCB = 40°$, $\angle OCD = \angle ODC = 55°$, and $\angle ODA = \angle OAD = x°$. Since the sum of the angles in any quadrilateral is 360°, $2(35) + 2(40) + 2(55) + 2x = 360$. Dividing throughout by 2 gives $35 + 40 + 55 + x = 180$ and hence $130 + x = 180$. Subtracting 130 from both sides gives $x = 50$. Then $\angle ODA = 50°$.



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29. Let Juan's age be J. Juan's sister is three years younger than him. Her age is $J - 3$. Juan's father is 27 years older than him. His age is $J + 27$. It is stated that Juan's age added to his sister's is half their father's age. Then

$$J + (J - 3) = \frac{1}{2}(J + 27)$$

Multiplying throughout by 2 gives $2J + 2(J - 3) = J + 27$, hence $2J + 2J - 6 = J + 27$, and hence $4J - 6 = J + 27$. Subtracting J and adding 6 to both sides gives $3J = 33$. Dividing both sides by 3 gives $J = 11$. Then Juan is 11 years old. (And his sister is 8 and his father is 38.)

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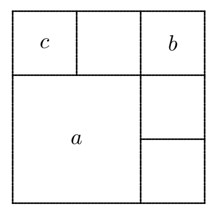
30. The pairs of numbers which differ by 3 are $\{1, 4\}$, $\{2, 5\}$, and $\{3, 6\}$. This means that 1 must not be adjacent to 4, 2 must not be adjacent to 5, and 3 must not be adjacent to 6. Suppose now a number, say a, is placed in the large square (see the figure to the right). There is exactly one number, call it b, that cannot be adjacent to a. It must be placed in the upper-right square as shown. Now let c be any other number. This may be placed in the upper left corner as shown. There is also exactly one number, call it d, which may not be placed adjacent to c. This number must be placed in one of the two boxes below b. The fifth number, call it e, may be placed between c and b. The last number must go in the last box. In summary, there are:



i) 6 possibilities for a, the number in the largest box;
ii) 1 possibility for b, the number in the opposite corner;
iii) 4 possibilities for c, the number in the upper left corner;
iv) 2 possibilities for the box which may contain d (the complement of c);
v) 2 possibilities for e, the number placed between c and b; and
vi) 1 possibility for placing the last number in the last box.

By the multiplication principle, there are $6 \times 1 \times 4 \times 2 \times 2 \times 1 = 96$ ways of filling the squares.