The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problem Set 2

1. For the number 112, the product of its digits is 2 and the sum of its digits is 4. For 209, the product of its digits is 0 and the sum of its digits is 11. For 312, the product of its digits is 6 and the sum of its digits is also 6. For 222, the product of its digits is 8 and the sum of its digits is 6. Finally, for 211 the product of its digits is 2 and the sum of its digits is 4. Among the given numbers, only 222 has the product of its digits greater than the sum of its digits.

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2. The numbers may be added and subtracted in reverse order to obtain $(55-54)+(53-52)+(51-50)+\cdots+(5-4)+(3-2)+1$. There are 27 expressions of the form $55-54$, $53-52$, and so on, up to $3-2$. Each of them is equal to 1. There is also one more term, namely 1, at the end. Then the original expression is equal to $1+1+1+\cdots+1$, where 1 occurs a total of 28 times. This is equal to 28.

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3. Since mother ate one-third of the pizza, she shared two-thirds of it between her children. But the fraction 2/3 is equivalent to 8/12. Mother shared 8/12 of the pizza among her children. She has 8 children.

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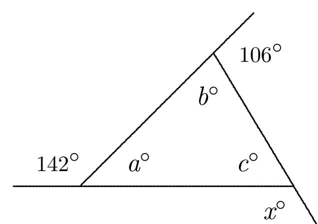
4. Let a, b, and c be the measures of the interior angles, as shown in the figure to the right. Since the three angles are formed by extending each side of the triangle, $142+a=180$, $106+b=180$, and $c+x=180$. From the first two equations, $a=38$ and $b=74$. Furthermore, $a+b+c=180$. Substituting for a and b, $38+74+c=180$. It follows that $c=68$. Since $x+c=180$ we have $x=112$.



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5. To end up with the smallest possible three-digit number, one first looks for the smallest possible digit for the hundreds place. In this case, it is 1. One then looks for the smallest digit to its right for the tens place. In this case it is 0. One finally looks for the smallest remaining digit on its right for the ones digit. In this case, it is 8. The smallest possible three-digit number is 108. It is formed by removing 4, 9, 2, and 5.

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6. Since 4 students study both languages, they should not be counted twice. The number who study at least one language is $17+13-4=26$.

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7. Since n is the product of the prime numbers less than 2014, $n=2\times3\times5\times7\times11\times\cdots\times2011$. (One may verify that 2011 is, indeed, prime.) Then n is a multiple of $2\times3\times5=30$, and hence is a multiple of 10. The last digit of n is 0.

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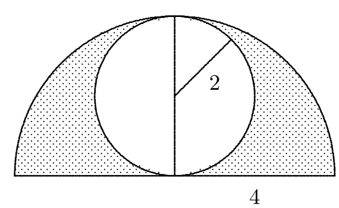
8. Since the radius of the inscribed circle is 2, its diameter is 4. Thus the radius of the outer semicircle is 4 as well. The area of this semicircle is $\frac{1}{2}\pi(4^2)=8\pi$. On the other hand, the area of the inscribed circle is $\pi(2^2)=4\pi$. Thus the area of the shaded region is $8\pi-4\pi=4\pi$. The proportion of the semicircle that is shaded is $4\pi/8\pi=1/2$.



9. First, consider the times of the form $ab\colon cd$, where a, b, c, and d are the numbers 2, 0, 1, and 2 in some order. There are 6 ways of replacing two of the letters with 2:

$$22\colon cd, \qquad 2b\colon 2d, \qquad 2b\colon c2, \qquad a2\colon 2d, \qquad a2\colon c2, \qquad ab\colon 22.$$

For each of these ways, we can choose the remaining letters to be 0 and 1 in two different ways. This gives $6 \times 2 = 12$ possibilities in all. They are:

$$\begin{array}{cccccc} 22\colon 01, & 20\colon 21, & 20\colon 12, & 02\colon 21, & 02\colon 12, & 01\colon 22, \\ 22\colon 10, & 21\colon 20, & 21\colon 02, & 12\colon 20, & 12\colon 02, & 10\colon 22. \end{array}$$

Each of these displays occurs on an electronic watch exactly once between 00:00 and 23:59. The answer is 12.

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10. The first and third lights flash together every 6 minutes. Thus they flash together after 6, 12, 18, 24, 30, 36, ... minutes. The second light flashes after 2.5, 5.0, 7.5, 10.0, 12.5, ... minutes. The first time all three lights flash together is after 30 minutes. So, after 9:00, the next time all three lights will flash together is 9:30 am.

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11. Each new pattern is formed by taking one tile (in the upper right corner) in the preceding pattern and replacing it with four smaller tiles. So, each new pattern has three tiles more than the old one. The first

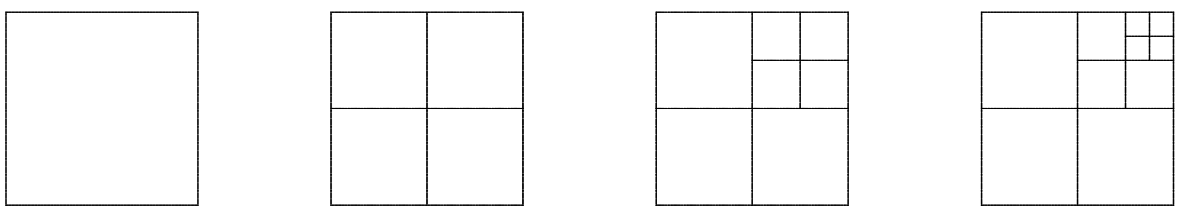

Figure 1 Figure 2 Figure 3 Figure 4

pattern has 1 tile, the second one has 4 tiles, the third has 7 tiles, the fourth has 10 tiles, the fifth has 13 tiles, and the sixth one has 16 tiles.

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12. We have

$$1 + \frac{1}{1 + \frac{1}{1+1}} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{\frac{2}{2} + \frac{1}{2}} = 1 + \frac{1}{\left(\frac{3}{2}\right)} = 1 + \frac{2}{3} = \frac{3}{3} + \frac{2}{3} = \frac{5}{3}.$$

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13. Since the perimeter of Square I is 16, each of its sides has length 4. Since the perimeter of Square II is 24, each of its sides has length 6. Then each side of Square III has length 10. Similarly, each side of Square IV has length 16. The perimeter of Square IV is $16 + 16 + 16 + 16 = 64$.

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14. The value of AB is $10A + B$. This is equal to $5(A + B) = 5A + 5B$. So, $10A + B = 5A + 5B$. Subtracting $5A + B$ from both sides gives $5A = 4B$. We know that A is a digit from 1 to 9, and B is a digit from 0 to 9. The possibilities for $5A$ and $4B$ are given by

$$\begin{array}{l} 5A\colon\ 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35,\ 40,\ 45 \\ 4B\colon\ 0,\ 4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 28,\ 32,\ 36 \end{array}$$

The only way $5A$ can be equal to $4B$ is if both are equal to 20. Then $A = 4$ and $B = 5$. The number AB is 45.

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15. From the middle column of Figure 15(a), we have $q \times 1 \times 4 = 1$. This means that $q = 1/4$. From the bottom row, $u \times 4 \times (1/8) = 1$. This means that $u = 2$. From the diagonal starting with p, $p \times 1 \times (1/8) = 1$.

p	q	r
s	1	t
u	4	1/8

Fig. 15(a)

8	1/4	r
s	1	t
2	4	1/8

Fig. 15(b)

Then $p = 8$. These values are shown in the table in Figure 15(b). From the top row there, $8 \times (1/4) \times r = 1$. Then $r = 1/2$. From the first column there, $8 \times s \times 2 = 1$. Then $s = 1/16$. Thus

$$r + s = \frac{1}{2} + \frac{1}{16} = \frac{8}{16} + \frac{1}{16} = \frac{9}{16}.$$

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16. The powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256, 512, ... Their last digits are 2, 4, 8, 6, 2, 4, 8, 6, 2, ... This repeats every 4 terms and continues forever. The 4th, 8th, 12th, and so on, terms are all 6. Eventually the 2012th term is 6, the 2013th term is 2, and the 2014th term is 4. The powers of 3 are 3, 9, 27, 81, 243, ... Their last digits are 3, 9, 7, 1, 3, ... This sequence also repeats every 4 terms and continues forever. The 4th, 8th, 12th, and so on, terms are all 1. Eventually the 2012th term is 1, the 2013th term is 3, and the 2014th term is 9. The last digit of the number $2^{2012} + 3^{2012}$ is the last digit of $4 + 9 = 13$. This is 3.

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17. Suppose abc is a three-digit number with c greater than a. If $a = 1$ then b could be any digit and c could be 2, 3, 4, 5, 6, 7, 8, or 9. There are 10 ways to choose b and, for each one, 8 ways to choose c. There are $10 \times 8 = 80$ numbers $1bc$ where c is greater than 1. If $a = 2$ then b could be any digit and c could be 3, 4, 5, 6, 7, 8, or 9. There are 10 ways to choose b and, for each one, 7 ways to choose c. There are $10 \times 7 = 70$ numbers $2bc$ where c is greater than 2. In the same way, there are 60 numbers $3bc$ where c is greater than 3, 50 numbers $4bc$ where c is greater than 4, and so on. This continues until there are 10 numbers $8bc$ where c is greater than 8. (In other words, $c = 9$). There are a total of $80 + 70 + 60 + \cdots + 10 = 360$ numbers abc with c greater than a.

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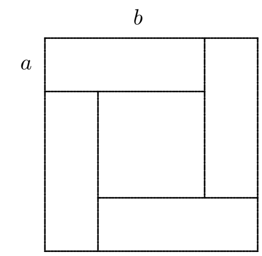
18. If a one-digit number divided into 1001 leaves a remainder of 5, the number divides 996 evenly. Then the number must be 1, 2, 3, 4, or 6. However, if 1001 is divided by 1, 2, 3, or 4, any remainder we obtain would have to be 3 or less. So, the one-digit number dividing 1001 is 6. When 6 divides 2014, the remainder is 4.

19. Let a be the length of a shorter side and b the length of a longer side of a rectangle. The side length of the outer square is $a + b$ and the side length of the inner square is $b - a$. Since the area of the outer square is 4 times that of the inner one,



$$\begin{aligned}(a+b)^2 &= 4(a-b)^2 \\ a^2+2ab+b^2 &= 4(a^2-2ab+b^2) \\ a^2+2ab+b^2 &= 4a^2-8ab+4b^2 \\ 0 &= 3a^2-10ab+3b^2\end{aligned}$$

Factoring the right side gives $(3a - b)(a - 3b) = 0$. Then either $3a - b = 0$ or $a - 3b = 0$. In the first case $a = b/3$, and in the second case $a = 3b$. Since a is the shorter side, $a = b/3$. Then the ratio of the length of the longer side to that of the shorter side is

$$\frac{b}{a} = \frac{b}{\left(\frac{b}{3}\right)} = \frac{b}{1}\left(\frac{3}{b}\right) = \frac{3}{1} = 3.$$

That is, the ratio is $3 : 1$.

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20. For the vendor to buy 360 chocolates at the supermarket, she must buy 45 boxes with 8 chocolates each. Since each box costs \$300, her total cost will be \$13,500. At the wholesale shop, she must buy 6 cases with 60 chocolates each. Since each case costs \$2,000, her total cost will be \$12,000. If she buys at the wholesale shop she will save $\$13,500 - \$12,000 = \$1,500$.

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21. Suppose one married couple is denoted by A_1 and A_2, another by B_1 and B_2, and the third by C_1 and C_2. The only way to form a three-person group with no married couple present is to choose one of the As, one of the Bs, and one of the Cs for the group. There are two ways to choose one of the As, two ways to choose one of the Bs, and two ways to choose one of the Cs. There are a total of $2 \times 2 \times 2 = 8$ ways of choosing a group like this. So, there are eight ways of forming a three-person group in which there will not be a married couple.

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22. First, $\dfrac{ab-b^2}{ab-a^2} = \dfrac{b(a-b)}{a(b-a)} = \dfrac{-b(b-a)}{a(b-a)} = -\dfrac{b}{a}$. Then

$$\frac{a^2-b^2}{ab} - \frac{ab-b^2}{ab-a^2} = \frac{a^2-b^2}{ab} - \left(-\frac{b}{a}\right) = \frac{a^2-b^2}{ab} + \frac{b}{a} = \frac{a^2-b^2}{ab} + \frac{b^2}{ab} = \frac{a^2}{ab} = \frac{a}{b}.$$

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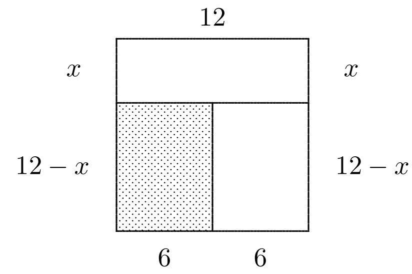
23. We know from the first turn that $2B + C = 10$ and from the second one that $2A + C = 22$. Adding these equations, $2A + 2B + 2C = 32$. Dividing throughout by 2 gives $A + B + C = 16$. Anna scored 16 points on her third turn.

24. The upper rectangle has width 12 and its height may be denoted by x, as shown in the figure to the right. (All dimensions are in centimetres.) The lower rectangles obviously have equal height and it is given that they have equal perimeter. This implies that they have equal width as well, which is 6 in both cases. Since the upper and lower rectangles have the same perimeter,



$$\begin{aligned} 12 + x + 12 + x &= (12 - x) + 6 + (12 - x) + 6 \\ 24 + 2x &= 36 - 2x \end{aligned}$$

Simplifying further gives $4x = 12$ and so $x = 3$. This means that each lower rectangle has height 9. Then the area of the shaded rectangle is $9 \times 6 = 54$.

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25. Let k be the number of children the Dobsons have, let $a_1, a_2, \ldots, a_k$ be their ages, and let m be the age of Mrs. Dobson. Since Mr. Dobson is 38 years old and the average age of the Dobson family is 18,

$$\frac{a_1 + a_2 + \cdots + a_k + m + 38}{k + 2} = 18.$$

Multiplying both sides by $k+2$ gives $a_1 + a_2 + \cdots + a_k + m + 38 = 18k + 36$. Subtracting 38 from both sides gives $a_1 + a_2 + \cdots + a_k + m = 18k - 2$. Since the average age of the family without Mr. Dobson is 14,

$$\frac{a_1 + a_2 + \cdots + a_k + m}{k + 1} = 14$$

Multiplying both sides by $k + 1$ gives $a_1 + a_2 + \cdots + a_k + m = 14k + 14$. It follows that $18k - 2 = 14k + 14$. Then $4k = 16$ and hence $k = 4$. The Dobsons have 4 children.

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26. We first determine the number of positive integers less than 1000 which are in fact multiples of 5 or 7 (or both). The 5-multiples less than 1000 are 5, 10, 15, 20, …, 995, and there are 199 such numbers in all. The 7-multiples less than 1000 are 7, 14, 21, 28, …, 994, and there are 142 such numbers in all. However, some numbers are multiples both of 5 and 7, and these are the multiples of 35. The 35-multiples less than 1000 are 35, 70, 105, 140, …, 980, and there are 28 such numbers in all. Then the number of positive integers less than 1000 which are multiples of 5 or 7 (or both) is $199 + 142 - 28 = 313$. The number of positive integers that are not multiples of 5 or 7 is $999 - 313 = 696$.

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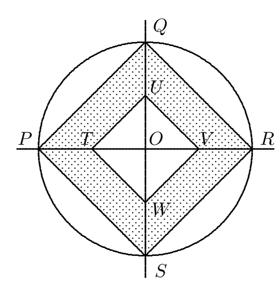
27. Let a be the radius of the circle. Then TO and OU are each $a/2$. The area of the triangle TOU is



$$\frac{1}{2}(TO)(OU) = \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{a^2}{8}.$$

It follows that the area of square $TUVW$, which is four times the area of $\triangle TOU$, is $a^2/2$. Similarly, the area of $\triangle POQ$ is $\frac{1}{2}(a)(a) = \frac{a^2}{2}$, and it the area of square $PQRS$ is $2a^2$. Then the shaded area is the area of the square $PQRS$ minus that of $TUVW$. This is

$$2a^2 - \frac{a^2}{2} = \frac{4a^2}{2} - \frac{a^2}{2} = \frac{3a^2}{2}.$$

On the other hand, the area of the outer circle is πa^2. Then the proportion of the area of this circle that is shaded is

$$\frac{3a^2/2}{\pi a^2} = \frac{3a^2}{2} \times \frac{1}{\pi a^2} = \frac{3a^2}{2\pi a^2} = \frac{3}{2\pi}.$$

28. Let h be the hypotenuse of the triangle and a the length of the other side. By the Pythagorean theorem, $a^2 + (\sqrt{60})^2 = h^2$ and hence $a^2 + 60 = h^2$. By the laws of algebra, $h^2 - a^2 = 60$ and so $(h+a)(h-a) = 60$. Since h and a are positive integers with $h > a$, $h + a$ and $h - a$ are also positive integers and $h + a > h - a$. Since $h + a$ and $h - a$ are factors of 60, the possibilities are:

$$h + a = 60, \quad h - a = 1; \qquad h + a = 20, \quad h - a = 3; \qquad h + a = 12, \quad h - a = 5;$$
$$h + a = 30, \quad h - a = 2; \qquad h + a = 15, \quad h - a = 4; \qquad h + a = 10, \quad h - a = 6.$$

When $h + a = 30$ and $h - a = 2$ then $h = 16$ and $a = 14$. This gives a right triangle with sides 14 and $\sqrt{60}$ and hypotenuse 16. When $h + a = 10$ and $h - a = 6$ then $h = 8$ and $a = 2$. This gives a right triangle with sides 2 and $\sqrt{60}$ and hypotenuse 8. In the other pairs of equations, the solutions for h and a are not integers. Thus there are two right triangles meeting our criteria: 14, $\sqrt{60}$, and 16, and 2, $\sqrt{60}$ and 8.

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29. The numbers added in the first step are $1 + 2$, $2 + 3$, and $3 + 1$. The sum of these numbers is $1+1+2+2+3+3 = 2(1+2+3) = 12$. Thus the sum of all the numbers on the circle after the first step is $6 + 12 = 18$. In general, suppose the numbers on the circle at some stage are a_1, a_2, …, a_k. Let s_k be the sum of these numbers, so that $s_k = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k$. Moving to the next stage, the sums of the neighboring numbers are $a_1 + a_2$, $a_2 + a_3$, …, $a_{k-1} + a_k$, and $a_k + a_1$. The sum of these new numbers is

$$(a_1 + a_2) + (a_2 + a_3) + \cdots + (a_{k-1} + a_k) + (a_k + a_1) = 2(a_1 + a_2 + \cdots + a_{k-1} + a_k) = 2s_k$$

When this is added to the numbers already on the circle, the total is $s_k + 2s_k = 3s_k$. We know that the sum of the numbers on the circle after the first step is 6, and the sum after the second step is 18. Taking three more steps, the sums are 54, 162, and 486, respectively. Thus, at the end, the sum of the 48 numbers on the circle is 486.

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30. Let O and P be the centres of the respective circles and let A and B be the points where the circles intersect. (See Figure 30(a) below.) Note that OA and OP are radii of the circle with centre O, and PO

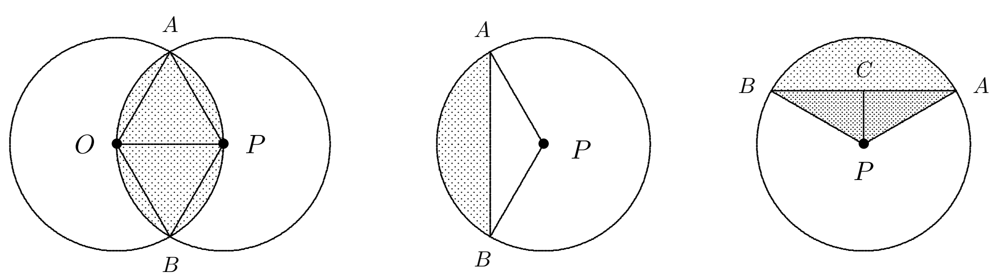

Fig. 30(a) Fig. 30(b) Fig. 30(c)

and PA are radii of the circle with centre P. Then $\triangle AOP$ is equilateral. Similarly, $\triangle BOP$ is equilateral. Thus $\angle APB = 60° + 60° = 120°$. The shaded region in Figure 30(a) consists of two sectors equal to the one shaded in Figure 30(b). By rotating one sector 90°, this may be viewed as the upper portion of an "ice-cream cone" as in Figure 30(c). The area of the entire ice-cream cone is 1/3 of the area of the circle with centre P. (This is because $\angle APB$ is 1/3 of an entire circle.) Its area is

$$\frac{1}{3}\,\pi(6^2) = \frac{1}{3}(36\pi) = 12\pi$$

To find the area of $\triangle BPA$, let C be the midpoint of BA. Then BPC and APC are each 30-60-90 triangles. It follows that $BP = PA = 6$, $CP = 3$, and $BC = AC = 3\sqrt{3}$. Then the area of BPA is

$$\frac{1}{2}(BA)(CP) = \frac{1}{2}\big(6\sqrt{3}\big)(3) = 9\sqrt{3}.$$

Thus the area of one sector (as in Figure 30(b)) is $12\pi - 9\sqrt{3}$. The total shaded area (as in Figure 30(a)) is $24\pi - 18\sqrt{3}$.