The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problem Set 3

1. Bob has an animal with fur but doesn't like cats. Then Bob has a dog. Also, Doug has a pet with four legs. Then Doug has a cat. Since Carl has a bird, he has a canary. This means that Alan has a goldfish. The only false sentence is "Doug has a dog".

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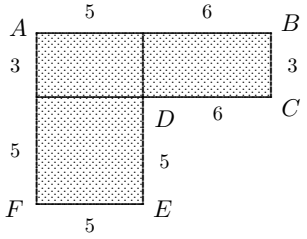
2. The figure may be divided into three rectangles as shown below. In the upper right corner, the area of the smaller rectangle is $6 \times 3 = 18$. In the upper left, the area of the rectangle is $5 \times 3 = 15$. In the lower left, the area of the rectangle is $5 \times 5 = 25$. The total shaded area is $18 + 15 + 25 = 58$.



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3. There are 3 branches in the rosebush, $3 \times 3 = 9$ roses in the rosebush, and $2 \times 3 \times 3 = 18$ bees in the rosebush.

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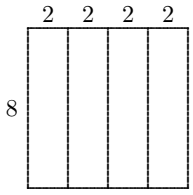
4. Since the area of the square is $64\,\text{cm}^2$, each side has length 8 cm. Then each smaller rectangle has width 2 cm and height 8 cm. Thus each smaller rectangle has perimeter (in centimetres) $2 + 8 + 2 + 8 = 20$.



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5. The plane has one row with four passengers and 23 rows with 6 passengers each. A total of $(23 \times 6) + (1 \times 4) = 138 + 4 = 142$ passengers are seated on the plane.

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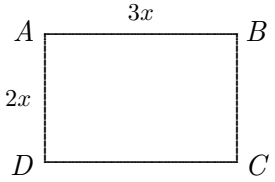
6. Since $AD : AB = 2 : 3$, there is some x such that $AD = 2x$ and $AB = 3x$. Since $ABCD$ has



area $150\,\text{cm}^2$,

$$(2x)(3x) = 150; \qquad 6x^2 = 150; \qquad x^2 = 25; \qquad x = 5.$$

Then the rectangle has height 10 cm and width 15 cm. Its perimeter is $10\,\text{cm} + 15\,\text{cm} + 10\,\text{cm} + 15\,\text{cm} = 50\,\text{cm}$.

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7. The total amount of sugar in the mixture, measured in litres, is $2(0.10)+3(0.15) = 0.20+0.45 = 0.65$. The mixture itself contains 5 litres of juice all together. The percentage of sugar in the juice is

$$\frac{0.65}{5} \times 100 = (0.13) \times 100 = 13.$$

The mixture contains 13% sugar.

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8. Let u be the unit of length which is one-third of the side of the square. Then each side of the square has length $3u$. The perimeter of the square, denoted by A, is given by $A = 12u$. The perimeter of the octagon, denoted by B is given by $B = 16u$. Then $\dfrac{A}{B} = \dfrac{12u}{16u} = \dfrac{12}{16} = \dfrac{3}{4}$.

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9. Suppose Martha sold m eggs on Monday and t eggs on Tuesday. On Wednesday she sold $m + t$ eggs. So, $m + t = 60$. On Thursday she sold $t + 60$ eggs. So, $t + 60 = 96$. Subtracting 60 from both sides gives $t = 36$. Martha sold 36 eggs on Tuesday.

Monday	Tuesday	Wednesday	Thursday	Friday
m	t	60	96	156

Since $m + t = 60$ we have $m + 36 = 60$. Subtracting 36 from both sides gives $m = 24$. Martha sold 24 eggs on Monday.

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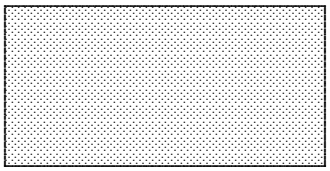
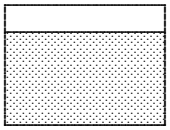
10. The side of the picture 160 cm long is scaled to fit exactly on a sheet of paper 40 cm long. Then the scaling factor is 1/4. The height of the picture, 80 cm, is also scaled by 1/4. Then the height of the picture's image is 20 cm. Since the sheet of paper is 30 cm high, the region not covered by the image is 10 cm high. It is also 40 cm long. Its area is $400\,\text{cm}^2$.



Picture: 80×160



Paper: 30×40

11. Suppose the seven-digit number is $abcdefg$. Since the sum of any four adjacent digits is 16, $a+b+c+d=16$. Since the sum of any 5 consecutive digits is 19, $a+b+c+d+e=19$. This means that $e=3$. Similarly, $b+c+d+e=16$ and $b+c+d+e+f=19$. Then $f=3$. Continuing, $c+d+e+f=16$ and $c+d+e+f+g=19$. Thus $g=3$. So far, the number $abcdefg$ has the form $abcd333$. On the other hand, $a+b+c+d+e=19$ and $b+c+d+e=16$. Then $a=3$. Similarly, $b+c+d+e+f=19$ and $c+d+e+f=16$. Then $b=3$. Finally, $c+d+e+f+g=19$ and $d+e+f+g=16$. Then $c=3$. Then the number $abcdefg$ has the form $333d333$. Finally, the sum of the first four digits is 16. Then $3+3+3+d=16$. This means that $9+d=16$ and so $d=7$. The number of dollars Eve sent the lawyer is $3,337,333$.

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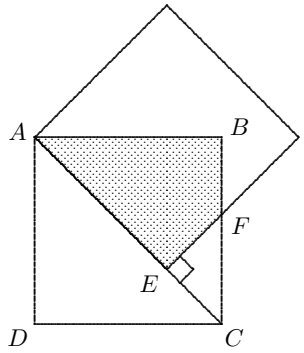
12. Let A, B, C, D, E, and F be the points shown on the figure below. The area of the shaded region is the area of triangle ABC minus the area of triangle EFC. The area of the square $ABCD$ is 1. Since the



triangles ABC and ADC are congruent by the side-side-side theorem, each of them has area $1/2$. To find the area of triangle EFC, note that $\angle FEC = 90^\circ$ and $\angle ECF = 45^\circ$. Then $\angle EFC = 45^\circ$ as well. Thus EFC is an isosceles triangle with $EF = EC$. Also, $AC = \sqrt{2}$ by the Pythagorean theorem and $AE = 1$. Then $EC = \sqrt{2}-1$. Thus $EF = \sqrt{2}-1$ as well. The area of $\triangle EFC$ is

$$\frac{1}{2}\big(\sqrt{2}-1\big)\big(\sqrt{2}-1\big) = \frac{1}{2}\big(2-2\sqrt{2}+1\big) = \frac{3}{2}-\sqrt{2}.$$

Finally, the area of the shaded region is

$$\frac{1}{2}-\left(\frac{3}{2}-\sqrt{2}\right) = \frac{1}{2}-\frac{3}{2}+\sqrt{2} = \sqrt{2}-1.$$

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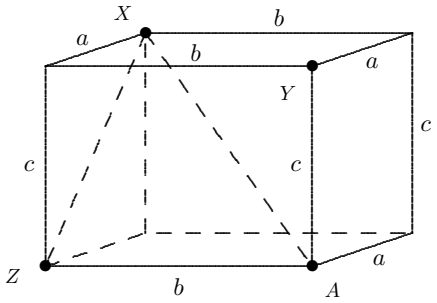
13. If a number has remainder 4 when divided by 5 then its last digit is either 4 or 9. If the number is odd then its last digit must be 9. If the number is also less than 3568 then it is one of these numbers (counting backwards):

$$3559,\ 3549,\ 3539,\ 3529,\ 3519,\ 3509,\ \ldots$$

The largest number on this list which has remainder 2 when divided by 3 is 3539. The sum of its digits is 20.

14. Let a, b, and c be the dimensions of the cuboid as shown. Since XZ is perpendicular to ZA, we have $(XZ)^2+(ZA)^2=(XA)^2$ by the Pythagorean theorem. Furthermore,



$(XZ)^2 = a^2 + c^2$ and $(ZA)^2 = b^2$. Thus $(XA)^2 = a^2 + b^2 + c^2$. Now note that $a^2 + b^2 = (XY)^2 = 64$, $a^2 + c^2 = (XZ)^2 = 55$, and $b^2 + c^2 = (YZ)^2 = 81$. Adding all three of these equations, $2a^2 + 2b^2 + 2c^2 = 64 + 55 + 81 = 200$. Thus $a^2 + b^2 + c^2 = 100$. It follows that $XA = \sqrt{100} = 10$.

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15. When the digits of a 3-digit number abc are put in reverse order the new number, cba, is larger if and only if $c > a$. The numbers abc for which $a < c$ (and b is unspecified) are as follows:

$1b2$,	$1b3$,	$1b4$,	$1b5$,	$1b6$,	$1b7$,	$1b8$,	$1b9$,
	$2b3$,	$2b4$,	$2b5$,	$2b6$,	$2b7$,	$2b8$,	$2b9$,
		$3b4$,	$3b5$,	$3b6$,	$3b7$,	$3b8$,	$3b9$,
			$4b5$,	$4b6$,	$4b7$,	$4b8$,	$4b9$,
				$5b6$,	$5b7$,	$5b8$,	$5b9$,
					$6b7$,	$6b8$,	$6b9$,
						$7b8$,	$7b9$,
							$8b9$,

There are 36 possibilities listed. For each one there are 10 choices for b. Then there are 360 such numbers in all.

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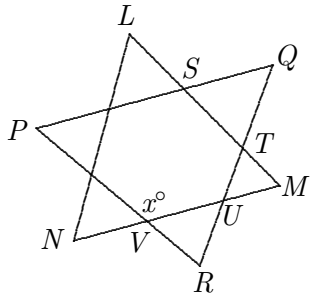
16. Let T, U, and V be the points of intersection as shown in the figure below. From the information given, the angles at Q, M, and R are each $60°$. Since the sum of the angles in



any triangle is $180°$, $\angle QTS = 65°$. By vertical angles, $\angle MTU = 65°$ as well. It follows that $\angle MUT = 55°$. By vertical angles again, $\angle RUV = 55°$. Thus $\angle RVU = 65°$ as well. Finally, since $\angle PVM$ and $\angle RVM$ are supplementary, $x + 65 = 180$. Then $x = 115$.