The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problem Set 4

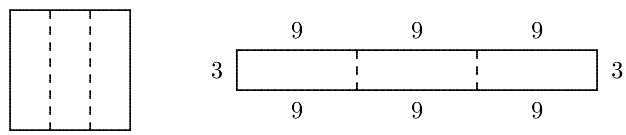
1. If $\dfrac{x/4}{2} = \dfrac{4}{x/2}$ then $\dfrac{x/4}{2/1} = \dfrac{4/1}{x/2}$. Since dividing by a fraction is the same as multiplying by its reciprocal,

$$\frac{x}{4} \times \frac{1}{2} = \frac{4}{1} \times \frac{2}{x} \quad \text{and so} \quad \frac{x}{8} = \frac{8}{x}.$$

Cross multiplying gives $x^2 = 64$. then $x = \pm 8$.

---

2. Since the square has area $81\,\text{cm}^2$, each side has length $9\,\text{cm}$. When the square is cut into three equal rectangles, each one is $3\,\text{cm} \times 9\,\text{cm}$. When the strips are placed end-to-end, the resulting rectangle is $27\,\text{cm} \times 3\,\text{cm}$. Its perimeter is $27\,\text{cm} + 3\,\text{cm} + 27\,\text{cm} + 3\,\text{cm} = 60\,\text{cm}$.



---

3. Let $a = 70$ and $b = 28$. If a is reduced by 20% the new number is 80% of 70. This is $(0.8)(70) = 56$. If b is reduced by 25% the new number is 75% of 28. This is $(0.75)(28) = 21$. The product of the reduced numbers is $56 \times 21 = 1176$.

---

4. Let the seats be numbered 1, 2, 3, 4, and 5, as shown in the figures below, where 1 is the seat number for Ashley. Since Brian is not sitting next to her, he is not in Seats 2 or 5.

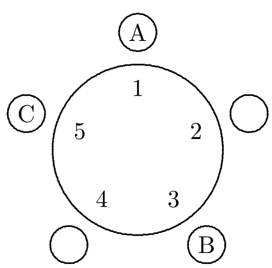


Figure 4(a)

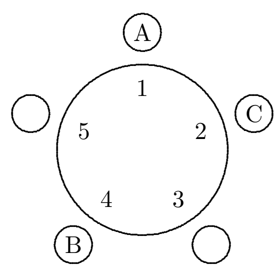
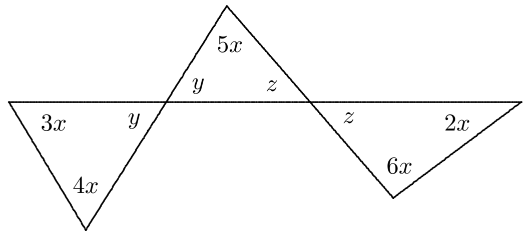


Figure 4(b)

Then he must be in Seats 3 or 4. Suppose he is in Seat 3 as in Figure 4(a). Since Carl is not next to Brian, he is not in Seats 2 or 4. Then Carl is in Seat 5. Since Dean is not next to Carl, he is not in Seat 4. Then Dean is in Seat 2. It follows that Elton is in Seat 4 and his neighbors are Brian and Carl.

Suppose now that Brian is in Seat 4 as in Figure 4(b) above. Since Carl is not next to Brian, he is not in Seats 3 or 5. Then Carl is in Seat 2. Since Dean is not next to Carl, Dean is not in Seat 3. Then Dean is in Seat 5. It follows that Elton is in Seat 3 and his neighbors once again are Brian and Carl. Thus in either case Elton is sitting next to Brian and Carl.

5. Let y and z be the angles shown in the diagram below. (Both angles labeled with y are equal because they are vertical angles, and similarly for the angles labeled with z.) From



the triangle on the left, $3x + 4x + y = 180°$. Then $y = 180° - 7x$. From the triangle on the right, $z+6x+2x = 180°$ and so $z = 180° - 8x$. From the triangle on the top, $5x+y+z = 180°$. Substituting for y and z gives $5x+(180°-7x)+(180°-8x) = 180°$. Simplifying gives $360°-10x = 180°$. Adding $10x$ to both sides gives $360° = 180° + 10x$. Subtracting $180°$ from both sides gives $180° = 10x$. dividing both sides by 10 gives $x = 18°$.

---

6. As stated in the question, the first four terms of the sequence are 1, 2, 3, and 6. The fifth term is $2 + 3 + 6 = 11$. The sixth term is $3 + 6 + 11 = 20$. The seventh term is $6 + 11 + 20 = 37$. The eighth term is $11 + 20 + 37 = 68$.

---

7. To begin, one can spend all of the $1500 to buy 15 boxes of chocolates. This gives a total of 15 coupons which may be redeemed for 5 more boxes. This gives another 5 coupons which may be redeemed for one box of chocolate with 2 coupons left over. Finally, the one box contains a coupon which may be redeemed with the other two for one last box of chocolate. The greatest number of boxes one can get for $1500 is $15 + 5 + 1 + 1 = 22$.

---

8. The original trapezoid is shown in Figure 8(a) below. The area of this trapezoid is equal to that of the outer rectangle in Figure 8(b). This is because the small triangle in the lower right

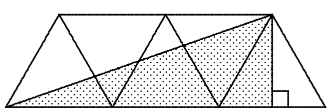


Figure 8(a)

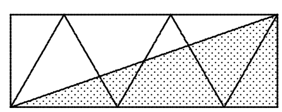


Figure 8(b)

corner in Figure 2(a) is congruent to the small triangle in the upper left corner in Figure 2(b). Since the shaded area in Figure 2(b) is obviously half the area of the rectangle, the shaded area in Figure 2(a) is half the area of the original trapezoid.

---

9. We may cross-multiply the given expression to obtain $3(2x - y) = 2(x + y)$ and hence $6x - 3y = 2x + 2y$. Subtracting $2x$ from both sides gives $4x - 3y = 2y$. Adding $3y$ to both sides gives $4x = 5y$. Dividing both sides by $4y$ gives $x/y = 5/4$.

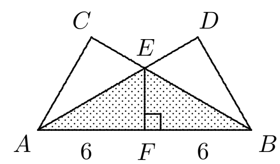
---

10. Let M and S be the current ages of Marco and Shawana. Since Marco was three times older than Shawana eighteen years ago, $M - 18 = 3(S - 18)$. Since Marco is now twice as old as Shawana,

$M = 2S$. Then, by substitution, $2S - 18 = 3(S - 18) = 3S - 54$. Adding 54 to the left and right sides of this equation gives $2S + 36 = 3S$. Subtracting $2S$ from both sides gives $S = 36$. Shawana is 36 years old.

---

11. Let E be the point where AD and CB intersect and let F be the point on AB such that $EF \perp AB$. Since $\angle ADB = 60°$ and $\angle BDA = 90°$, it follows that $\angle DAB = 30°$. Similarly,



$\angle CBA = 30°$. Thus $\triangle EAB$ is isosceles and $EA = EB$. Furthermore, since $\angle EAF = 30°$ and $\angle AFE = 90°$, it follows that $\angle AEF = 60°$. That is, AFE is a 30-60-90 triangle. Thus $EF : AF : AE = 1 : \sqrt{3} : 2$. In the same way, BFE is a 30-60-90 triangle. Thus the triangles AEF and BEF are similar. Since $AE = EB$ (and $EF = EF$), they are in fact congruent. Then $AF = FB = 6$. It follows that $EF : 6 = 1 : \sqrt{3}$. That is, $\dfrac{EF}{6} = \dfrac{1}{\sqrt{3}}$ and so $EF = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$. Then the area of $\triangle AEB$ is $\frac{1}{2}(12)(2\sqrt{3}\,) = 12\sqrt{3}$.

---

12. Listed from least to greatest, the lowly numbers are:

1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222,
2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222.

The sum of the 8th and 9th numbers is $1222 + 2111 = 3333$.

---

13. Given \$80 to start with, a shopper may purchase 2 mangos, 1 mango, or no mangos at all. We consider each case separately.

If a shopper purchases 2 mangos there will be \$20 remaining. This may be spent on 1 apple or 2 bananas. There are two possibilities in this case.

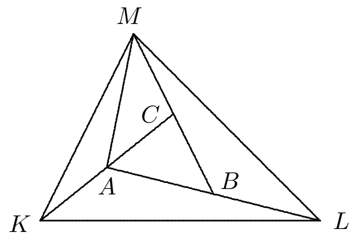
If a shopper purchases 1 mango there will be \$50 remaining. This may be spent on 2 apples and 1 banana, 1 apple and 3 bananas, or 5 bananas. There are three possibilities in this case.

If a shopper does not purchase a mango he or she will spend \$80 on apples and bananas. This may be spent on 4 apples, 3 apples and 2 bananas, 2 apples and 4 bananas, 1 apple and 6 bananas, or 8 bananas. There are five possibilities in this case.

In summary, considering the three cases together, there are $2 + 3 + 5 = 10$ ways of purchasing fruit in the market.

---

14. Join A and M with a segment as shown in the figure below. Since $BC = CM$ the triangles ACM and ACB have equal bases. Since the triangles both meet at A, they also have equal



(perpendicular) heights. Then ABC and AMC both have the same area, which is 1. Now observe that since $AB = BL$ the triangles MAB and MBL have equal bases. Since both meet at M, they also have equal heights. Thus they both have the same area, which is 2. By reasoning in the same way, one shows that the triangles MCK and KAL also have area 2. Then the area of KLM is the combined areas of ABC, MBL, MCK, and KAL. This is $1 + 2 + 2 + 2 = 7$.

---

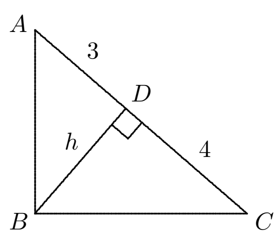
15. Let the five boys' weights be a, b, c, d, and e, respectively. The weights of each possible pair of boys are

$$\begin{array}{cccc} a+b & b+c & c+d & d+e \\ a+c & b+d & c+e & \\ a+d & b+e & & \\ a+e & & & \end{array}$$

If we add these expressions, the sum of the weights of all pairs of boys is $4a + 4b + 4c + 4d + 4e$. On the other hand, the sum of the weights of all pairs of boys is also $90 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 100 + 101 = 956$. Then $4a + 4b + 4c + 4d + 4e = 956$. Dividing throughout by 4 gives $a + b + c + d + e = 239$. The total weight of all five boys is 239 kg.

---

16. Let h be the length of BD. The triangles ADB and ABC are similar. This is because $\angle ADB = \angle ABC = 90^\circ$ and $\angle DAB = \angle BAC$. Thus we also have $\angle ABD = \angle BCD$. Since



corresponding sides of similar triangles are in the same proportion, $AD/DB = BD/DC$. Thus $3/h = h/4$. Cross multiplying gives $h^2 = 12$. Then $h = 2\sqrt{3}$. The base AC of $\triangle ABC$ is 7 and the height BD is $2\sqrt{3}$. The area of $\triangle ABC$ is

$$\frac{1}{2}(7)(2\sqrt{3}) = 7\sqrt{3}.$$