The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problem Set 5

1. Note that $a+3b+c$ is a multiple of 3 if and only if $a+c$ is a multiple of 3. Then we must consider numbers abc such that $a+c$ is a multiple of 3. Since we cannot have $a=0$, the possibilities are:

$$\begin{array}{lll} a=1,\ c=2,\ 5,\ 8; & a=4,\ c=2,\ 5,\ 8; & a=7,\ c=2,\ 5,\ 8; \\ a=2,\ c=1,\ 4,\ 7; & a=5,\ c=1,\ 4,\ 7; & a=8,\ c=1,\ 4,\ 7; \\ a=3,\ c=0,\ 3,\ 6,\ 9; & a=6,\ c=0,\ 3,\ 6,\ 9; & a=9,\ c=0,\ 3,\ 6,\ 9; \end{array}$$

There are 30 possibilities in all. For each one, there are 10 choices for b. So, there are 300 numbers abc such that $a+3b+c$ is a multiple of 3.

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2. Let d be the distance from 1/4 to x_1. Then d is also the distance from x_1 to x_2, from x_2 to x_3, and from x_3 to 1/3. The total distance from 1/4 to 1/3 is $4d$. But the distance from 1/4 to 1/3 is also

$$\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{1}{12}.$$

Then $4d=1/12$. This means that $d=1/48$. It follows that $x_1=\frac{1}{4}+d=\frac{1}{4}+\frac{1}{48}=\frac{12}{48}+\frac{1}{48}=\frac{13}{48}$.

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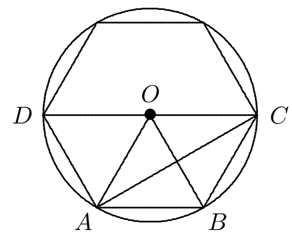
3. Let D be the point opposite C, as shown in the figure below. The lines CD and AB are parallel. This is because $\angle COB=\angle OBA=60°$. Hence they form equal "Z-angles" between DC and AB. Then the



triangles AOB and ACB have the same area. (They have the same base, AB, and equal heights.) But triangle AOB has base 1 and height $\sqrt{3}/2$. Its area is $\sqrt{3}/4$. Then the area of $\triangle ABC$ is $\sqrt{3}/4$.

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4. For any N to be considered, $24N$ must be a multiple of 36. This happens when $N=3$, 6, 9, 12, ... Also, $36N$ must be a multiple of 24. This happens when $N=2$, 4, 6, 8, 10, 12, ... The values of N satisfying both conditions at once are $N=6$, 12, 18, ... The smallest of these values is $N=6$. This value also satisfies the condition that 24×36 is a multiple of N. So, 6 is the smallest possible value for N.

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5. Let ab be a fascinating number. Its value is $10a+b$. When its digits are reversed, the new number has value $10b+a$. This is 75% larger than the original number when

$$10b+a=(10a+b)+\frac{75}{100}(10a+b)=(10a+b)+\frac{3}{4}(10a+b)=\frac{7}{4}(10a+b)$$

Multiplying throughout by 4 gives $40b + 4a = 70a + 7b$. Subtracting $7b$ from both sides gives $33b + 4a = 70a$. Subtracting $4a$ from both sides gives $33b = 66a$. Dividing both sides by 33 gives $b = 2a$. The two-digit numbers ab with $b = 2a$ are 12, 24, 36, and 48. These are all fascinating, as one may verify. So, there are four such numbers in all.

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6. Let abc be a 3-digit number such that b is the average of a and c. It is not possible that $b = 0$ because then $a = 0$ and $c = 0$ as well (and 000 is not a 3-digit number.) So we consider the cases with $b = 1, 2, 3, \ldots, 9$. The possibilities are as follows:

- $b = 1$: Then $abc = 111$ or 210;
- $b = 2$: Then $abc = 123$, 222, 321, or 420;
- $b = 3$: Then $abc = 135$, 234, 333, 432, 531, or 630;
- $b = 4$: Then $abc = 147$, 246, 345, 444, 543, 642, 741, or 840;
- $b = 5$: Then $abc = 159$, 258, 357, 456, 555, 654, 753, 852, or 951;
- $b = 6$: Then $abc = 369$, 468, 567, 666, 765, 864, or 963;
- $b = 7$: Then $abc = 579$, 678, 777, 876, or 975;
- $b = 8$: Then $abc = 789$, 888, or 987;
- $b = 9$: Then $abc = 999$,

There are $2 + 4 + 6 + 8 + 9 + 7 + 5 + 3 + 1 = 45$ balanced numbers in all.

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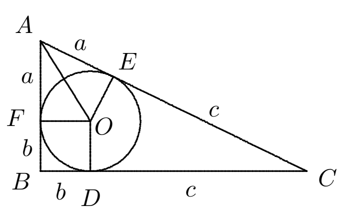
7. Let O be the centre of the circle, and let D, E, and F be the points on BC, CA, and AB, respectively, where the circle is tangent to the triangle. Join O with F, with D, with E, and with A. Since F and E are points of tangency, $\angle AFO = \angle AEO = 90°$. Also, $FO = OE$ because



both are radii of the circle. By the Pythagorean theorem (used twice), $(AF)^2 = (AO)^2 - (FO)^2 = (AO)^2 - (OE)^2 = (AE)^2$. Since AF and AE are both positive, $AF = AE$. One proves similarly that $BF = BD$ and $CD = CE$. Let $a = AF = AE$, $b = BF = BD$, and $c = CD = CE$. We have the following three equations in three unknowns:

$$\begin{cases} a + b = 3 \\ b + c = 4 \\ a + c = 5 \end{cases}$$

The unique solution is $a = 2$, $b = 1$, and $c = 3$. In particular, $b = 1$. But $OFBD$ is a square because $\angle OFD = \angle FBD = \angle BDO = 90°$ and $FB = BD$. Thus $OF = b = 1$. The circle has radius 1.

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8. For $n = 1, 2, 3, \ldots, 99$,

$$\frac{1}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \; \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{(n+1) - n} = \frac{\sqrt{n+1} - \sqrt{n}}{1} = \sqrt{n+1} - \sqrt{n}$$

Thus $\frac{1}{\sqrt{1}+\sqrt{2}} = \sqrt{2}-\sqrt{1}, \quad \frac{1}{\sqrt{2}+\sqrt{3}} = \sqrt{3}-\sqrt{2}, \quad \frac{1}{\sqrt{3}+\sqrt{4}} = \sqrt{4}-\sqrt{3}$, and so on. Then

$$\begin{aligned} x &= \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} \\ &= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{100}-\sqrt{99}) \end{aligned}$$

Writing these expressions in reverse order gives

$$x = (\sqrt{100}-\sqrt{99}) + (\sqrt{99}-\sqrt{98}) + \cdots + (\sqrt{4}-\sqrt{3}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{2}-\sqrt{1})$$

All terms except the first and last ones cancel. Then $x = \sqrt{100}-\sqrt{1} = 10-1 = 9$.