The 2015 Jamaican Mathematical Olympiad

Solutions for Practice Problem Set 7

1. From the first three squares, $3 + A + B = 15$. Thus $A + B = 12$. From the second, third, and fourth squares, $A + B + C = 15$ as well. Then $12 + C = 15$ and so $C = 3$. From the fourth, fifth,

3	A	B	C	D	8	E	F	G

3	4	8	3	4	8	3	4	8

and sixth squares, $3 + D + 8 = 15$ and so $D = 4$. From the fifth, sixth, and seventh squares, $4 + 8 + E = 15$ and so $E = 3$. Finally, from the sixth, seventh, and eighth squares, $8 + 3 + F = 15$ and so $F = 4$. The completed table is shown on the right above.

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2. The entries in the first column are 9, 17, 25, 33, ... These are the numbers of the form $8n + 1$. Eventually (*i.e.*, when $n = 125$), the number 1001 occurs in the first column. The number 1000 occurs in Column 2.

	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
⋮	⋮	⋮	⋮	⋮

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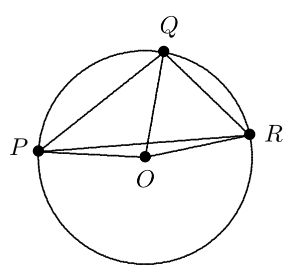
3. Since OP, OQ, and OR are radial segments, they are equal in length. Then the triangles OPQ, OPR, and OQR are all isosceles with $\angle OPQ = \angle OQP = 40°$, $\angle OPR = \angle ORP = 5°$, and



$\angle OQR = \angle ORQ$. Since the sum of the angles in any triangle is $180°$, $\angle POR = 180° - 5° - 5° = 170°$. Similarly, $\angle POQ = 180° - 40° - 40° = 100°$. Then $\angle QOR = \angle POR - \angle POQ = 170° - 100° = 70°$. Finally, $\angle QOR + \angle OQR + \angle ORQ = 180°$. Substituting for $\angle QOR$ and $\angle ORQ$ gives $70° + 2\angle OQR = 180°$. Subtracting $70°$ from both sides gives $2\angle OQR = 110°$. Dividing both sides by 2 gives $\angle OQR = 55°$.

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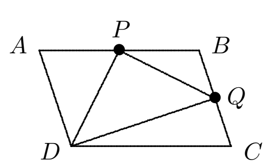
4. First, the area of $\triangle DAB$ is half of the area of $ABCD$. Then the area of $\triangle DAB$ is 12. Also, since P is the midpoint of AB, the area of $\triangle DAP$ is half the area of $\triangle DAB$. Then the area



of $\triangle DAP$ is 6. By reasoning in the same way, one sees that the area of $\triangle DQC$ is 6 as well. Similarly, the area of $\triangle PQB$ is half the area of $\triangle PCB$, and this is half the area of $\triangle ACB$, and this is half the area of $ABCD$. Then the area of $\triangle PQB$ is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 24 = 3.$$

Finally, the area of $\triangle DPQ$ is the area of $ABCD$ minus the areas of $\triangle DAP$, $\triangle DCQ$, and $\triangle PQB$. So, the area of $\triangle DPQ$ is $24 - (6 + 6 + 3) = 24 - 15 = 9$.

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5. Car I can possibly be in the first, second, third, or fourth position in the train. (It cannot be in the last position because then Car II could not be behind it.) We consider each possibility in turn.

If Car I is in the first position there are $4 \times 3 \times 2 \times 1 = 24$ ways of arranging the other four cars. (Each time, Car II will be behind Car I). This gives 24 possibilities in this case.

If Car I is in the second position any car except Car II may be in the first position. This gives 3 possibilities. For each one, there are $3 \times 2 \times 1 = 6$ ways of arranging the last three cars. There are $3 \times 6 = 18$ possibilities in this case.

If Car I is in the third position there are 3 choices for the car in the first position and two for the one in the second position. For each of these pairs of choices, there are $2 \times 1 = 2$ ways of arranging the last two cars. Thus there are $6 \times 2 = 12$ possibilities in this case.

Finally, if Car I is in the fourth position then car II must be in the fifth position. The other three cars will fill the first three positions and there are $3 \times 2 \times 1 = 6$ ways of doing this.

Putting these cases together, there are $24 + 18 + 12 + 6 = 60$ ways of arranging the cars so that Car I is closer to the engine than Car II is.

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6. First, Maria can decide which days she will eat the apples on. If a stands for apple and x stands for another piece of fruit, she could eat the apples in any of the following ways:

$a,\ a,\ x,\ x,\ x$	$x,\ a,\ a,\ x,\ x$	$x,\ x,\ a,\ a,\ x$	$x,\ x,\ x,\ a,\ a$
$a,\ x,\ a,\ x,\ x$	$x,\ a,\ x,\ a,\ x$	$x,\ x,\ a,\ x,\ a$	
$a,\ x,\ x,\ a,\ x$	$x,\ a,\ x,\ x,\ a$		
$a,\ x,\ x,\ x,\ a$			

There are 10 ways of choosing the days to eat an apple on. Once this is done, there are three more days to choose fruits for. Suppose b stands for a banana and m for a mango. The fruits on these days could be in the order of b, b, m, or b, m, b, or m, b, b. There are 3 possibilities in all. So there are 10 ways of deciding which days to eat apples on, and for each one there are 3 ways of deciding on the other fruits. There are $10 \times 3 = 30$ ways for Maria to eat her fruit.

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7. We seek solutions to the equation $2^{2x} - 3^{2y} = 55$. We may rewrite this as $(2^x)^2 - (3^y)^2 = 55$. Since the left side is the difference of two squares, we have $(2^x + 3^y)(2^x - 3^y) = 55$. Since x and y are positive integers, 2^x and 3^y are both positive integers. Thus $2^x + 3^y$ and $2^x - 3^y$ are factors

of 55. Furthermore, $2^x + 3^y > 2^x - 3^y$. Since the factors of 55 are 1, 5, 11, and 55, the possibilities are:

$$\begin{cases} 2^x + 3^y = 55 \\ 2^x - 3^y = 1 \end{cases} \qquad \begin{cases} 2^x + 3^y = 11 \\ 2^x - 3^y = 5 \end{cases}$$

In the first case, adding equations gives $2(2^x) = 56$ and so $2^x = 28$. However, there is no integer x solving this equation. In the second case, adding equations gives $2(2^x) = 16$ and hence $2^x = 8$. This has solution $x = 3$. Substituting into either equation gives $y = 1$ as well. Then $(3, 1)$ is the only pair of positive integers (x, y) such that $2^{2x} - 3^{2y} = 55$.

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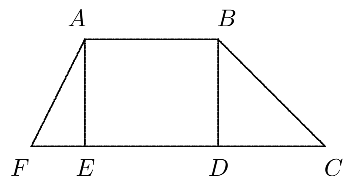
8. Since the area of $ABDE$ is 80, we have $(AB)(AE) = 80$. Since the area of $ABCF$ is 128, we have $\frac{1}{2}(AB + CF)(AE) = 128$. Multiplying both sides by 2 gives $(AB + CF)(AE) = 256$. Thus



$(AB)(AE) + (CF)(AE) = 256$. Substituting for $(AB)(AE)$ gives $80 + (CF)(AE) = 256$. Subtracting 80 from both sides gives $(CF)(AE) = 176$. Then $\dfrac{AB}{CF} = \dfrac{(AB)(AE)}{(CF)(AE)} = \dfrac{80}{176} = \dfrac{5}{11}$. That is, $AB : CF = 5 : 11$.

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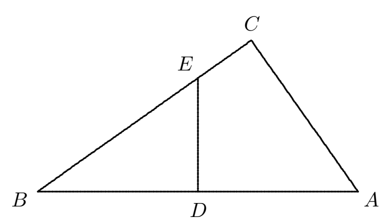
9. Since $\triangle ABC$ is a right triangle, $(AC)^2 + (BC)^2 = (AB)^2$. Substituting for AB and AC gives $12^2 + (BC)^2 = 20^2$ and so $144 + (BC)^2 = 400$. Subtracting 144 from both sides gives $(BC)^2 = 256$.



Then $BC = 16$. Since $\angle C$ is a right angle, the area of $\triangle ABC$ is equal to $\frac{1}{2}(16)(12) = 96$. To find the area of $\triangle BED$, observe that the triangles BED and ABC are similar. This is because both have a right angle and both share the angle at B. Since the sum of the angles in any triangle is 180°, it follows that $\angle BED = \angle BAC$ as well. Since corresponding sides of similar triangles are in the same proportion, $ED/BD = CA/CB$. Substituting for BD, CA, and CB gives $ED/10 = 12/16$. Cross-multiplying gives $16(ED) = 120$. Dividing both sides by 16 gives $ED = 120/16 = 15/2$. It follows that the area of $\triangle BED$ is equal to $\frac{1}{2}(10)\left(\frac{15}{2}\right) = \frac{75}{2}$. Finally, the area of $ADEC$ is equal to the area of $\triangle ABC$ minus the area of $\triangle BED$. This is $96 - \frac{75}{2} = \frac{117}{2}$.

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10. Let

$$S_1 = \{1, 4, 7, 10, 13, 16, 19, 22, 25\},$$
$$S_2 = \{2, 5, 8, 11, 14, 17, 20, 23\},$$
$$S_3 = \{3, 6, 9, 12, 15, 18, 21, 24\}.$$

Then S_1 consists of all the numbers in S with remainder 1 when divided by 3, S_2 consists of all the numbers with remainder 2, and S_3 consists of all the numbers divisible by 3. Note that the sum of any two numbers in S_1 is not divisible by 3. Similarly, the sum of any two numbers in S_2 is not divisible by 3. However, the sum of any number in S_1 and any number in S_2 is divisible by 3. Let T be a subset of S with the property that the sum of any two elements in T is not divisible by 3. Then T can contain all the elements in S_1 or all the elements in S_2, but it cannot contain elements from both subsets. Furthermore, suppose T contains all the elements in S_1. Then it can also contain one element from S_3. Similarly, if T contains all the elements in S_2 it may also contain one element from S_3. However, T cannot contain two elements from S_3. For example if T contains 6 and 15 then it contains $6 + 15 = 21$. But 21 is a multiple of 3. Then the possibilities are that:

a) T contains all elements in S_1 and one element from S_3, or

b) T contains all elements in S_2 and one element from S_3.

Since (a) gives a set with more elements, T contains 10 elements. For example, we could have $T = \{1, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$

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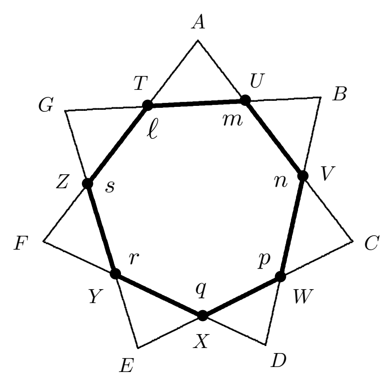
11. Let ℓ, m, n, p, q, r, and s be the interior angles at the vertices T, U, V, W, X, Y, and Z, respectively. Then $\angle ATU = 180° - \ell$ and $\angle AUT = 180° - m$. Since the sum of



the angles in any triangle is 180°, $\angle A + \angle ATU + \angle AUT = 180°$. Substituting for $\angle AUT$ and $\angle ATU$ gives $\angle A + (180° - \ell) + (180° - m) = 180°$. Simplifying gives $\angle A + 180° = \ell + m$. Reasoning in a similar way, we obtain

$$\angle B + 180° = m + n; \qquad \angle C + 180° = n + p; \qquad \angle D + 180° = p + q;$$
$$\angle E + 180° = q + r; \qquad \angle F + 180° = r + s; \qquad \angle G + 180° = s + \ell.$$

Adding these seven equations together gives

$$\begin{aligned}&\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G + 1260° \\ &\quad = (\ell + m) + (m + n) + (n + p) + (p + q) + (q + r) + (r + s) + (s + \ell) \\ &\quad = 2(\ell + m + n + p + q + r + s)\end{aligned}$$

But $\ell + m + n + p + q + r + s$ is the sum of the interior angles of the heptagon $TUVWXYZ$. This is equal to $(7 - 2)180° = 5(180°) = 900°$. Thus

$$\begin{aligned}&\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G \\ &\quad = 2(\ell + m + n + p + q + r + s) - 1260° = 2(900°) - 1260° = 540°\end{aligned}$$