THE UNIVERSITY OF THE WEST INDIES, MONA

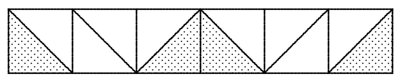
The 2016 Jamaican Mathematical Olympiad

Solutions for Grades 7 and 8

1. The fraction $1/2$ is in lowest terms. Regarding the other fractions, $2/6 = 1/3$, $3/12 = 1/4$, $4/20 = 1/5$, and $5/30 = 1/6$. The fraction with the smallest value is $5/30 = 1/6$.

---

2. A diagonal of a square divides it into 2 equal triangles. If each of the 6 squares were divided by a diagonal line, there would be 12 equal triangles in all. Since 4 of them are shaded, $4/12 = 1/3$ of the rectangle's area is shaded.



---

3. Since 75% of the boys registered for the trip, 12 boys registered. Since 50% of the girls registered for the trip, 5 girls registered. Then $12 + 5 = 17$ students registered for the trip.

---

4. The pharmacist has 3,200 millilitres of medicine. If he pours it into bottles holding 80 millilitres each, he will fill 40 bottles.

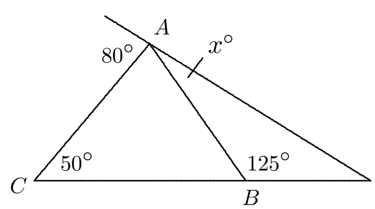
---

5. We have

$$(1-2)-(3-4)-(5-6)-(7-8)-(9-10)-(11-12)$$
$$= -1-(-1)-(-1)-(-1)-(-1)-(-1) = -1+1+1+1+1+1 = 4$$

---

6. Label the vertices of one of the triangles A, B, and C, as shown below. The measure of $\angle ABC$ is $55°$. This is because $\angle ABC$ is the supplement of an angle measuring $125°$. The measure of $\angle CAB$



is $75°$. This is because the sum of the angles in triangle ABC is $180°$. Finally, the angle marked $80°$, angle CAB, and the angle marked $x°$ together make a straight angle. Thus $80° + 75° + x° = 180°$. Solving for x gives $x = 25$.

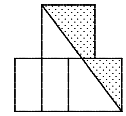
---

7. Since any 2 students gave different answers, they could not both be telling the truth. So only 1 student told the truth, or else nobody did. If nobody told the truth then nobody was ready for the exam. In this case, Al would have told the truth. Since this is a contradiction, it is impossible that nobody told the truth. Exactly 1 student told the truth (and this student was Beth). Then 1 student was ready for the test.

---

8. The next palindrome after 2002 is 2112. So the smallest number that can be added to 2002 to give another palindrome is 110.

9. The combined area of the shaded and unshaded regions is 3 The unshaded region consists of a large right triangle and a small rectangle immediately to its left. The base of the triangle is 3/2 and its height is 2. Its area is $(1/2)(3/2)(2) = 3/2$. The rectangle has base 1/2 and height 1. Its area is 1/2. The total area of the unshaded region is $(1/2) + (3/2) = 2$. The area of the shaded region is $3 - 2 = 1$.



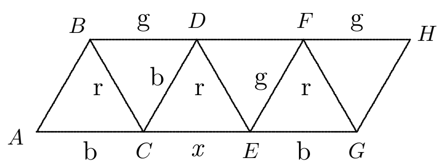
---

10. Let x_1 and x_2 be the two numbers whose average is 12. Then $(x_1 + x_2)/2 = 12$. Multiplying both sides by 2 gives $x_1 + x_2 = 24$. Let x_3, x_4, and x_5 be the numbers whose average is 7. Then $(x_3 + x_4 + x_5)/3 = 7$. Multiplying both sides by 3 gives $x_3 + x_4 + x_5 = 21$. The average of all five numbers is

$$\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{(x_1 + x_2) + (x_3 + x_4 + x_5)}{5} = \frac{24 + 21}{5} = \frac{45}{5} = 9.$$

---

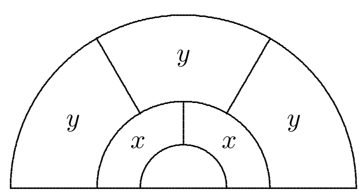
11. Let the vertices of the triangles be labeled A, B, ... , H, as shown in the figure below. If BC is blue then triangle ABC would have two blue sides. If BC is green then triangle BCD would have two



green sides. So BC is red. Since BD is green, CD must be blue. Also, FG must be red. (If it were green then FHG would have two green sides, and if it were blue then FGE would have two blue sides.) Since EG is blue, EF must be green. If side DE is green then triangle DFE would have two green sides. If DE is blue then triangle DEC would have two blue sides. So DE is red. Finally, triangle DEC has a blue side, a red side, and a side marked x. The side marked x is green.

---

12. The concentric semicircles have radii 1, 2, and 4, respectively. The full circles that contain them have areas $\pi(1^2) = \pi$, $\pi(2^2) = 4\pi$, and $\pi(4^2) = 16\pi$, respectively. The area of the middle band, containing



the two regions marked with x and two other regions equal in size and shape, is $4\pi - \pi = 3\pi$. Each region marked with x has one-fourth of this area. So $x = 3\pi/4$. The area of the outer band, containing the three regions marked with y and three regions equal in size and shape, is $16\pi - 4\pi = 12\pi$. Each region marked with y has one-sixth of this area. So $y = 2\pi$. Then $x : y = 3\pi/4 : 2\pi = 3\pi : 8\pi = 3 : 8$.

---

13. Since all scores for the pegs are odd numbers, the total score for three rings will be an odd number. The least possible score will be 3 points, occuring when all three rings land on Peg A. The greatest possible score is 15, occuring when all three rings land on Peg C. Any odd number between 3 and 15 is also possible. For example, to generate scores of 5, 7, 9, 11, and 13, the rings could land on pegs A, A, B, A, A, C, A, B, C, A, C, C, and B, C, C, respectively. The possible point totals are 3, 5, 7, 9, 11, 13, and 15. There are 7 such totals in all.

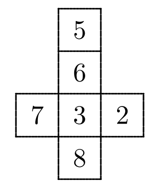
14. First,

$$w + \cfrac{1}{x + \cfrac{1}{y}} = w + \cfrac{1}{\cfrac{xy+1}{y}} = w + \frac{y}{xy+1} = \frac{97}{19} = 5\frac{2}{19}$$

Also, $xy + 1 > y$ because x is a positive integer and hence at least 1. Thus $y/(xy+1)$ is a fraction between 0 and 1. Since w is a positive integer, we must have $w = 5$ and $y/(xy+1) = 2/19$. One can prove that $y = 2$ and $xy + 1 = 19$. Substituting for y gives $x = 9$. Thus $w = 5$, $x = 9$, and $y = 2$, and $w + x + y = 16$.

---

15. When the net is folded into a cube, the numbers 6 and 8 will be on opposite sides. So it is not possible for Janice to obtain the product $8 \times 7 \times 6 = 336$. However, the sides with 8, 7, and 5 will meet at a corner. The largest product Janice can get is $8 \times 7 \times 5 = 280$.



---

16. Figure 16(b) below shows an enlarged view of two adjacent triangles from the octagon. The side length of the interior octagon is equal to DC. In turn, this is equal to $AC - AD$. By the Pythagorean theorem,

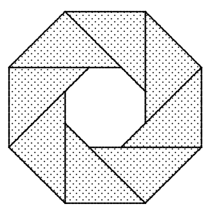


Figure 16(a)

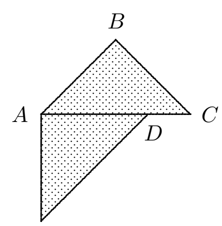


Figure 16(b)

$(AC)^2 = (AB)^2 + (BC)^2 = 1^2 + 1^2 = 2$. (Note that AB is the measure of a side of the outer octagon.) Then $AC = \sqrt{2}$. Length AD is also 1 because $AD = AB$. Then $DC = AC - AD = \sqrt{2} - 1$.

---

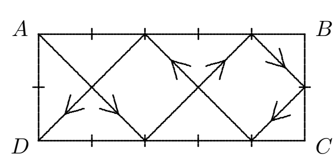
17. We have $4^{2015} + 4^{2015} + 4^{2015} + 4^{2015} = 4\left(4^{2015}\right) = 4^1 4^{2015} = 4^{2016}$.

---

18. We have $a \times b = 10,000 = 10^4 = 2^4 \times 5^4$. Since a is a factor of $2^4 \times 5^4$, its prime factorization is of the form $2^p \times 5^q$, where $0 \le p, q \le 4$. It is not possible that $p = 0$ and $q = 0$. Otherwise, $a = 1$ and so $b = 10,000$. But then b is divisible by 10. So either $p \ge 1$ or $q \ge 1$. If both $p \ge 1$ and $q \ge 1$, then a would be a multiple of 10. Since this is not so, either $p \ge 1$ and $q = 0$, or $q \ge 1$ and $p = 0$. In other words, a is either a power of 2 or a power of 5. Reasoning in the same way, b is either a power of 2 or a power of 5. Since $a \times b = 2^4 \times 5^4$, either $a = 2^4$ and $b = 5^4$, or vice-versa. In either case, $a + b = 16 + 625 = 641$.

---

19. As shown below, $AB = DC = 5$ and $AD = BC = 2$. Starting from A, the ball will bounce 5 times before it reaches D.



20. Since m is even, $3m$ is even. Also, $4n$ is always even. So, $3m+4n$ is even. Since m is even, $5mn$ is even. Since m is even, $5m$ is even and $6n$ will always be even. So, $5m+6n$ is even. Since m is even, m^3 is even and m^3n^3 is even. Finally, since n is odd, $3n$ is odd. Since m is even, $m+3n$ is odd. Since an odd times an odd is odd, $(m+3n)^2$ is odd. Then $(m+3n)^2$ is the only odd expression.

---

21. Let a and b be the number of marbles that Al and Ben have, respectively. If Al gave Ben 6 of his marbles, Al would have $a-6$ marbles and Ben would have $b+6$ marbles. Since they would have the same number of marbles, $a-6=b+6$. Adding 6 to both sides gives $a=b+12$. If Al gave Ben half of his marbles, Al would have $a/2$ marbles and Ben would have $b+a/2$ marbles. Since Ben would have 8 marbles more than Al, $b+a/2=a/2+8$. Subtracting $a/2$ from both sides gives $b=8$. Then $a=8+12=20$. Together, Al and Ben have $20+8=28$ marbles.

---

22. Let M be the number in the corner opposite to N. In the diagonal containing 6, 12, and 24, the product of its entries is 1728. So the product of the entries in any row, column, or diagonal is 1728. In particular, $12MN=1728$. Dividing both sides by 12 gives $MN=144$. Thus

N		24
	12	
6		M

$$N \in \{1,2,3,4,6,8,9,12,16,18,24,36,48,72,144\}.$$

However, if $N=144$ or $N=1$ then the box cannot be completed. If $N=144$ then the product of the numbers in the top row would be at least 3456. If $N=1$ then $M=144$, and the product of the numbers in the third column would be least 3456. Also, if $N=48$ or $N=3$ the box cannot be completed. If $N=48$ then the number in the middle of the top row would be 1.5. But all numbers in the small boxes must be integers. Similarly, if $N=3$ then $M=48$, and the number in the middle of the third column would be 1.5. Finally, if $N=16$ or $N=9$ the box cannot be completed. If $N=16$ then the number in the middle of the top row would be 4.5. Similarly, if $N=9$ then $M=16$, and the number in the middle of the third column would be 4.5. So we have

$$N \in \{2,4,6,8,12,18,24,36,72\}.$$

Furthermore, if Sherika uses any of these numbers then the box can be successfully completed. The cases with $N=2$, 4, 6, 8, and 12 are shown in Figures 21(a)-21(e) below. The cases for $N=18$, 24, 36, and 72 may also be obtained from the first four boxes. For example, the case with $N=72$ may be obtained by transposing the box in Figure 21(a) about the diagonal with entries 24, 12, and 6. In summary, Sherika has 9 choices for N that allow her to complete the box.

2	36	24
144	12	1
6	4	72

Figure 21(a)

4	18	24
72	12	2
6	8	36

Figure 21(b)

6	12	24
48	12	3
6	12	24

Figure 21(c

8	9	24
36	12	4
6	16	18

Figure 21(d)

12	6	24
24	12	6
6	24	12

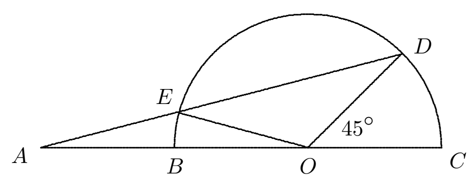
Figure 21(e)

23. We have

$$\begin{aligned}
S &= \frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}} + \frac{\sqrt{3}-\sqrt{2}}{\sqrt{6}} + \frac{\sqrt{4}-\sqrt{3}}{\sqrt{12}} + \cdots + \frac{\sqrt{9}-\sqrt{8}}{\sqrt{72}} \\
&= \left(\frac{\sqrt{2}}{\sqrt{2}} - \frac{\sqrt{1}}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{6}}\right) + \left(\frac{\sqrt{4}}{\sqrt{12}} - \frac{\sqrt{3}}{\sqrt{12}}\right) + \cdots + \left(\frac{\sqrt{9}}{\sqrt{72}} - \frac{\sqrt{8}}{\sqrt{72}}\right) \\
&= \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{9}}\right) \\
&= 1 + \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) + \left(-\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}}\right) + \cdots + \left(-\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8}}\right) - \frac{1}{3} \\
&= 1 - \frac{1}{3} = \frac{2}{3}.
\end{aligned}$$

---

24. Join E and O as in the figure below. Since $AE = OC$ and $EO = OC$ (both lines are radii of the semicircle), $AE = EO$. Then the triangle AEO is isosceles with $\angle EAO = \angle EOA$. Let x be the mea-



sure of these angles. Since the sum of the angles in any triangle is $180°$, $\angle AEO = 180° - 2x$. Since $\angle OED$ is the supplement of $\angle AEO$, $\angle OED = 2x$. Since $OE = OD$ (both lines are radii of the semicircle), triangle EOD is isosceles with $\angle ODE = \angle OED = 2x$. Finally, since $\angle AOD$ is the supplement of $\angle DOC$, $\angle AOD = 135°$. Summing the angles in triangle AOD, we have $x + 2x + 135° = 180°$. Simplifying gives $3x = 45°$. Dividing both sides by 3 gives $x = 15°$. The measure of $\angle DAO$ is $15°$.

---

25. Let N be a four-digit number which is divisible by 12 and whose digits sum to 6. We know that N is divisible by 12 if and only if it is divisible by 3 and by 4. (This is because 3 and 4 have no common factors.) Since the digits of N sum to 6, N is certainly divisible by 3. To be divisible by 4 as well, the last two digits of N, regarded as a two-digit number by itself, must be divisible by 4. So, the last two digits of N must be 00, 04, 12, 20, 32, or 40. We consider each possibility in turn.

If the last two digits of N are 00 then N could be 6000, 5100, 4200, 3300, 2400, or 1500. There are 6 such numbers in all. If the last two digits of N are 04 then N could be 2004 or 1104. There are 2 such numbers in all. If the last two digits of N are 12 then N could be 3012, 2112, or 1212. There are 3 such numbers in all. If the last two digits of N are 20 then N could be 4020, 3120, 2220, or 1320. There are 4 such numbers in all. If the last two digits of N are 32 then N must be 1032. There is only 1 number in this case. If the last two digits of N are 40 then N could be 2040 or 1140 There are 2 such numbers in all. In summary, there are $6 + 2 + 3 + 4 + 1 + 2 = 18$ numbers which are divisible by 12 and whose digits sum to 6.