THE UNIVERSITY OF THE WEST INDIES, MONA

The 2016 Jamaican Mathematical Olympiad

Solutions for Grades 9, 10, and 11

1. We have $x = 2w$ and $y = 2x = 2(2w) = 4w$. Then $w + x + y = w + 2w + 4w = 7w$.

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2. If two positive integers have a sum of 6, they are either 1 and 5, 2 and 4, or 3 and 3. Since the product of the integers is 5, the numbers are 1 and 5. So the sum of their squares is $1^2 + 5^2 = 1 + 25 = 26$.

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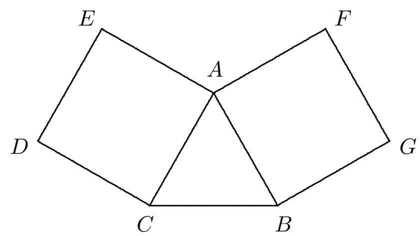
3. Since $AEDC$ and $AFGB$ are squares, $\angle EAC = \angle FAB = 90°$. Since triangle ABC is equiliateral, $\angle CAB = 60°$. Summing the angles around point A, $\angle EAF + 90° + 60° + 90° = 360°$. Thus $\angle EAF + 240° = 360°$. Subracting $240°$ from both sides gives $\angle EAF = 120°$.



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4. The lions, snakes, and parrots have 20 wings between them. Since the parrots have all the wings, there are 10 parrots in the zoo. The animals also have 60 legs between them. Since the parrots have 20 legs, the lions have the other 40 legs. There are 10 lions in the zoo. Finally, the animals have 40 heads between them. Since 10 belong to the parrots and 10 to the lions, 20 belong to snakes. There are 20 snakes in the zoo.

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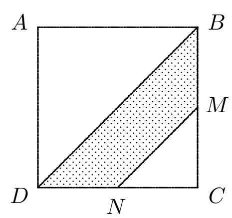
5. The area of square $ABCD$ is $2 \times 2 = 4$. Diagonal BD divides this square into two congruent triangles. So, the area of triangle BCD is 2. Since N is the midpoint of DC, $NC = 1$. Similarly,



$CM = 1$. Then the area of triangle MCN is $\frac{1}{2}(1)(1) = \frac{1}{2}$. Finally, the area of the shaded region is the area of triangle BCD minus the area of triangle MCN. This is $2 - \frac{1}{2} = \frac{3}{2}$.

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6. We have

$$(a + b + c - d) + (a + b - c + d) + (a - b + c + d) + (-a + b + c + d)$$
$$= 2a + 2b + 2c + 2d = 2(a + b + c + d) = 2(1 + 10 + 100 + 1{,}000) = 2(1111) = 2222.$$

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7. Note that 2002 is a palindrome and the next one after that is 2112. Since 2036 is between these two numbers, the smallest number that can be added to 2036 to make a palindrome is $2112 - 2036 = 76$.

8. Since 3 is odd, all of its powers are odd. So, given that a is a positive integer, 3^a is odd. Also, given that b is an odd integer, $b-1$ is even. So, no matter which positive integer c may be, $(b-1)c$ is also even. Finally, $3^a + (b-1)c$ is the sum of an odd term and an even term. This sum will be odd for all choices of c.

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9. There cannot be more than one true statement on the card. These is because each statement contradicts the other statements on the card, and two contradictory statements cannot both be true.

On this card exactly one statement is false. On this card exactly two statements are false. On this card exactly three statements are false. On this card exactly four statements are false.

So, at most one statement on the card is true. If none of them were true then all four statements, including the last one, would be false. But then the last statement would also be true. Since this is impossible, exactly one statement is true. Thus three statements on the card are false (and the third one is true).

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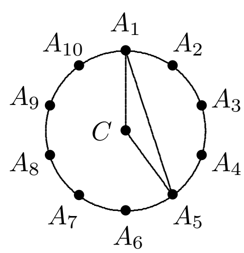
10. The arcs A_1A_2, A_2A_3, ... , $A_{120}A_1$ divide the circle into 10 equal parts. Since equal arcs subtend equal central angles, angles A_1CA_2, A_2CA_3, and so on, divide the circle into 10 equal angles. Each



one has measure 36°. It follows that $\angle A_1CA_5$ has measure 144°. Also, triangle A_1CA_5 is isosceles with $A_1C = CA_5$. Then the angles CA_1A_5 and CA_5A_1 are equal and together have measure $180° - 144° = 36°$. Each one has measure 18°. In particular, $\angle A_1A_5C = 18°$.

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11. The numbers between 100 and 1000 divisible by 3 are 102, 105, 108, ... , 999. These correspond exactly (by dividing by 3) to the numbers 34, 35, 36, ..., 333. There are 300 such numbers in all. The numbers between 100 and 1000 divisible by 7 are 105, 112, 119, ... , 994. These correspond exactly (by dividing by 7) to the numbers 15, 16, 17, ..., 142. There are 128 such numbers in all. Howeveer, some numbers are divisible both by 3 and by 7, and they have been counted twice. These are the multiples of 21. The multiples of 21 between 100 and 1000 are 105, 126, 147, ... , 987. These correspond exactly (by dividing by 21) to the numbers 5, 6, 7, ..., 47. There are 43 such numbers in all. Then there are $300 + 128 - 43 = 385$ numbers between 100 and 1000 which are divisible by 3 or 7 (or both).

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12. We have

$$\left(\sqrt{\sqrt{9}+\sqrt{1}}\right)^4 = \left(\sqrt{3+1}\right)^4 = \left(\sqrt{4}\right)^4 = 2^4 = 16.$$

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13. Let abc be a three-digit number. Its value is $100a + 10b + c$. The values of bca and cab are $100b + 10c + 1$ and $100c + 10a + b$, respectively. The sum $abc + bca + cab$ has value

$$100(a+b+c) + 10(a+b+c) + (a+b+c) = 111(a+b+c)$$

This is a three-digit number of the form ddd if and only if $a + b + c \leq 9$. Since $a < b < c$, the possible numbers are 123, 124, 125, 126, 134, 135, and 234. There are 7 such numbers in all.

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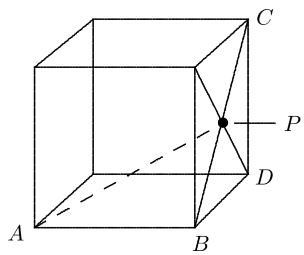
14. Label three vertices of the cube B, C, and D, as shown below. Then BCD is a right triangle with hypotenuse BC. By the Pythagorean theorem, $2^2 + 2^2 = (CB)^2$. Thus $(CB)^2 = 8$ and so



$CB = \sqrt{8} = 2\sqrt{2}$. Since the diagonals of a square bisect each other, PB is half the measure of CB. Thus $PB = \sqrt{2}$. Finally, PBA is a right triangle with hypotenuse AP. By the Pythagorean theorem, $(AP)^2 = (AB)^2 + (BP)^2 = 2^2 + \left(\sqrt{2}\right)^2 = 4 + 2 = 6$. Then $AP = \sqrt{6}$.

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15. We seek pairs of positive integers (a, b) with $a + b \leq 100$ such that $a + b^{-1} = 13\left(a^{-1} + b\right)$. That is, $a + 1/b = 13(1/a + b)$. Finding common denominators and adding on both sides gives

$$\frac{ab + 1}{b} = \frac{13(ab + 1)}{a}$$

Dividing both sides by $ab + 1$ gives $1/b = 13/a$. Cross-multiplying gives $a = 13b$. The pairs of positive integers (a, b) with $a + b \leq 100$ solving the original equation are $(13, 1)$, $(26, 2)$, $(39, 3)$, $(52, 4)$, $(65, 5)$, $(78, 6)$, and $(91, 7)$. There are 7 such pairs in all.

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16. Let ab be a two-digit number. Its value is $10a + b$. When its digits are reversed the new number is ba. The value of the new number is $10b + a$. Then the value of ba is 9 more than the value of ab exactly when $10b + a = 10a + b + 9$. Subtracting $a + b$ from both sides gives $9b = 9a + 9$. Dividing throughout by 9 gives $b = a + 1$. The two-digit numbers ab with $b = a + 1$ are 12, 23, 34, 45, 56, 67, 78, and 89. There are 8 such numbers in all.

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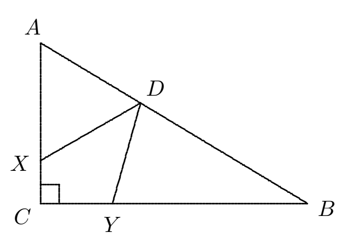
17. Since the sum of the angles in any triangle is $180°$, $\angle XAD + \angle ADX + \angle AXD = 180°$. Since $\triangle ADX$ is isosceles with $AX = AD$, $\angle AXD = \angle ADX$. Then $\angle XAD + 2\angle ADX = 180°$. Solving for



$\angle ADX$ gives $\angle ADX = 90° - \frac{1}{2}\angle XAD$. In the same way, $\angle BDY = 90° - \frac{1}{2}\angle DBY$. Since $\angle ADX + \angle XDY + \angle YDB = 180°$, we have

$$\left(90° - \frac{1}{2}\angle XAD\right) + \angle XDY + \left(90° - \frac{1}{2}\angle DBY\right) = 180°$$

Solving for $\angle XDY$ gives $\angle XDY = \frac{1}{2}(\angle XAD + \angle DBY)$. But $\angle XAD + \angle DBY = 90°$ because these angles are the acute angles in a right triangle. Then $\angle XDY = \frac{1}{2}(90°) = 45°$.

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18. Let the first six terms of the progression be a, ar, ar^2, ar^3, ar^4, and ar^5. Then $a + ar = 9$ and $a + ar + ar^2 + ar^3 + ar^4 + ar^5 = 189$. Thus

$$\begin{aligned}189 = (a + ar) + (ar^2 + ar^3) + (ar^4 + ar^5) &= (a + ar) + r^2(a + ar) + r^4(a + ar)\\ &= (1 + r^2 + r^4)(a + ar).\end{aligned}$$

Since $a + ar = 9$, we have $1 + r^2 + r^4 = 21$. Subtracting 21 from both sides gives $r^4 + r^2 - 20 = 0$. Factoring the left side gives $(r^2 - 4)(r^2 + 5) = 0$. Then $r^2 = 4$ or $r^2 = -5$. Since $r^2 = -5$ is impossible, we have $r^2 = 4$. Then $r = \pm 2$. Substituting $r = 2$ into the equation $a + ar = 9$ gives $a = 3$. In this case, the progression is 3, 6, 12, 24, 48, 96, ... The sum of its first four terms is 45. Substituting $r = -2$ into the equation $a + ar = 9$ gives $a = -9$. In this case, the progression is -9, 18, -36, 72, -144, 288, ... The sum of its first four terms is 45 once again. So, in either case the sum of the first four terms of the progression is 45.

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19. We have

$$\begin{aligned}999{,}999{,}999 \times 123{,}456{,}789 &= (1{,}000{,}000{,}000 - 1)123{,}456{,}789\\ &= 123{,}456{,}789{,}000{,}000{,}000 - 123{,}456{,}789 = 123{,}456{,}788{,}876{,}543{,}211\end{aligned}$$

None of the digits in this product are a 9.

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20. Let a, b, c, and d be the four positive integers. If two of them are equal then two of the subtotals would be the same. For example, if $a = b$ then the sums $a + c + d$ and $b + c + d$ would be equal. But the four given subtotals are distinct. So a, b, c, and d are also distinct. Relabeling, if necessary, let $a < b < c < d$. We have

$$a + b + c < a + b + d < a + c + d < b + c + d.$$

Then $a + b + c = 180$, $a + b + d = 197$, $a + c + d = 208$, and $b + c + d = 222$. Adding these four equations gives $3a + 3b + 3c + 3d = 807$. Dividing throughout by 3 gives $a + b + c + d = 269$. However, we've seen that $a + b + c = 180$. Subtracting this equation from the last one gives $d = 89$. (Proceeding in this way, one also determines that $a = 47$, $b = 61$, and $c = 72$.

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21. Let AG be an altitude for $\triangle ABC$ (see Figure 21(a) below). The triangles ACG and ABG are congruent by the angle-angle-side theorem. (We have $\angle ACG = \angle ABG$ because $\triangle ABC$ is isosceles,

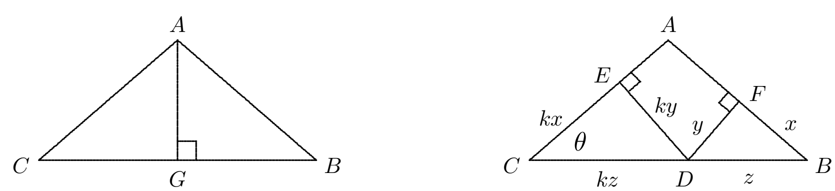

Figure 21(a) Figure 21(b)

$\angle AGC = \angle AGB = 90°$, and the side AG is common to both triangles.) Then $CG = GB$, and both lines have length 20. By the Pythagorean theorem, $(AG)^2 + (GC)^2 = (AC)^2$. Substituting for AC and CG gives $(AG)^2 + 20^2 = 25^2$. Then $(AG)^2 = 25^2 - 20^2 = 625 - 400 = 225$. Thus $AG = 15$. Let $\theta = \angle ACG$. Then $\cos\theta = 20/25 = 4/5$ and $\sin\theta = 15/25 = 3/5$. Observe now that triangles CED and BFD are similar. This is because $\angle ECD = \angle FBD$ and $\angle CED = \angle BFD = 90°$. It follows that $\angle EDC = \angle FDB$ as well. Since these triangles are similar, there is some k such that $CE = k(FB)$, $ED = k(DF)$, and $CD = k(BD)$. Let

$x = FB$, $y = DF$, and $z = DB$ as in Figure 21(b). Then $CE = kx$, $ED = ky$, and $CD = kz$ as well. Furthermore, $ky/kz = \sin\theta = 3/5$. Solving for y gives $y = 3z/5$. Then

$$ED + DF = ky + y = (k+1)y = \frac{3}{5}(k+1)z = \frac{3}{5}(kz+z) = \frac{3}{5}(CD+DB) = \frac{3}{5}(40) = 24.$$

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22. Given that $x_1 = 1$ and $x_2 = 3$, we have

$$x_3 = x_2 - x_1 = 3 - 1 = 2, \qquad x_4 = x_3 - x_2 = 2 - 3 = -1,$$
$$x_5 = x_4 - x_3 = (-1) - 2 = -3, \quad \text{and} \quad x_6 = x_5 - x_4 = (-3) - (-1) = -2$$

Continuing in this way, $x_7 = 1$, $x_8 = 3$, and so on. The terms of the sequence repeat in cycles of 6, and the sequence $\{s_1, , s_2, s_3, \ldots\}$ is the sequence

$$\{1, 3, 2, -1, -3, -2, 1, 3, 2, -1, -3, -2, 1, 3, 2, \ldots\}$$

If we add any six consecutive terms in this sequence the sum will be 0. It follows that the sum of the first 96 terms of this sequence is 0. The sum of the 97th, 98th, 99th, and 100th terms is $1 + 3 + 2 + (-1) = 5$.

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23. Let w be the amount of water and a the amount of acid, each measured in ounces, in the original mixture. Adding 1 ounce of water gives $a + w + 1$ ounces of mixture. Since this is 20% acid,

$$\frac{a}{a+w+1} = \frac{20}{100} = \frac{1}{5}.$$

Cross-multiplying gives $5a = a + w + 1$. Solving for w gives $w = 4a - 1$. Adding 1 ounce of acid gives $a + w + 2$ ounces of a new mixture. Since this is $33\frac{1}{3}\%$ acid,

$$\frac{a+1}{a+w+2} = \frac{33\frac{1}{3}}{100} = \frac{1}{3}.$$

Cross-multiplying gives $3a + 3 = a + w + 2$. Solving for w gives $w = 2a + 1$. Equating the two expressions for w gives $4a - 1 = 2a + 1$. Subtracting $2a$ from both sides gives $2a - 1 = 1$. Solving for a gives $a = 1$. Substituting into an expression for w gives $w = 3$. The original mixture consisted of 1 ounce of acid and 3 ounces of water. The original mixture was 25% acid.

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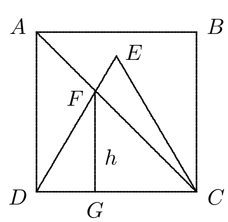
24. Let FG be an altitude for $\triangle DFC$ and let h be its height. Since $\angle FDG = 60\circ$ and $\angle DGF = 90°$, $\triangle DFG$ is a 30-60-90 triangle. It follows that $DG : GF : FD = 1 : \sqrt{3} : 2$. Thus $DG = h/\sqrt{3}$. Also,



$\angle GCF = 45°$ and $\angle CGF = 90°$. It follows that $\angle GFC = 45°$ as well. Thus $\triangle GFC$ is isosceles with $GF = GC = h$. Then $DC = DG + GC = \dfrac{h}{\sqrt{3}} + h = \dfrac{(1+\sqrt{3})h}{\sqrt{3}}$. On the other hand, $DC = \sqrt{1+\sqrt{3}}$ because DC is a side of the square. Equating these expressions for DC gives

$$\frac{(1+\sqrt{3})h}{\sqrt{3}} = \sqrt{1+\sqrt{3}}$$

Multiplying both sides by $\sqrt{3}$ gives $(1+\sqrt{3})h = \sqrt{3}\sqrt{1+\sqrt{3}}$. Dividing both sides by $1+\sqrt{3}$ gives

$$h = \frac{\sqrt{3}\sqrt{1+\sqrt{3}}}{1+\sqrt{3}} = \frac{\sqrt{3}\sqrt{1+\sqrt{3}}}{\left(\sqrt{1+\sqrt{3}}\right)^2} = \frac{\sqrt{3}}{\sqrt{1+\sqrt{3}}}$$

Finally, the area of $\triangle DFC$ is $\dfrac{1}{2}(DC)h = \dfrac{1}{2}\sqrt{1+\sqrt{3}}\left(\dfrac{\sqrt{3}}{\sqrt{1+\sqrt{3}}}\right) = \dfrac{\sqrt{3}}{2}$.

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25. Let $A = \{1, 2, 3, \ldots, 50\}$, and define subsets $A_1, A_2, \ldots, A_7$ of A as follows:

$$A_1 = \{1, 8, 15, 22, 29, 36, 43, 50\}, \quad A_2 = \{2, 9, 16, 23, 30, 37, 44\}, \quad A_3 = \{3, 10, 17, 24, 31, 38, 45\}$$
$$A_4 = \{4, 11, 18, 25, 32, 39, 46\}, \quad A_5 = \{5, 12, 19, 26, 33, 40, 47\}, \quad A_6 = \{6, 13, 20, 27, 34, 41, 48\}$$
$$A_7 = \{7, 14, 21, 28, 35, 42, 49\}$$

Subset A_1 consists of the numbers in A that have remainder 1 when divided by 7. Subset A_2 consists of the numbers in A that have remainder 2 when divided by 7. Similarly, A_3, A_4, A_5, and A_6, consist of the numbers with remainder 3, 4, 5, and 6, respectively, when divided by 7. Subset A_7 consists of the 7-multiples in A. (These have remainder 0 when divided by 7.)

If any number in A_1 is added to a number in A_6, the resulting sum will be a multiple of 7. (Try it.) Similarly, if any number in A_2 is added to a number in A_5, or any number in A_3 is added to a number in A_4, or any two numbers in A_7 are added, their sum will be a multiple of 7. Furthermore, these are the only ways of adding two numbers in A and getting a sum which is a multiple of 7. In other words, suppose two numbers in A are added and their sum is a multiple of 7. Then one of them is in A_1 and the other is in A_6, or one is in A_2 and the other is in A_5, or one is in A_3 and the other is in A_4, or both numbers are in A_7. (Try it again.)

Let S be a subset of A with the property that no pair of distinct numbers in S has a sum divisible by 7. To make S as large as possible, we may include all elements in A_1, A_2, and A_3. If we do so, however, we may not include any elements in A_4, A_5, or A_6. Finally, we may include 1 element from A_7, but no more than that. The largest number of elements S may have is $8 + 7 + 7 + 1 = 23$.