

# THE 2010 JAMAICAN MATHEMATICAL OLYMPIAD

PRESENTED BY THE UNIVERSITY OF THE WEST INDIES  
IN COLLABORATION WITH STERLING ASSET MANAGEMENT LTD

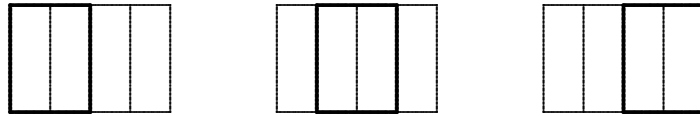
## QUALIFYING ROUND SOLUTIONS FOR GRADES 7 AND 8

- 1) Let  $x$  be the number. Then

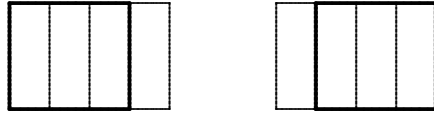
$$\frac{3}{4}x = 48; \quad \frac{3x}{4} = \frac{48}{1}; \quad 3x = 48(4) = 192; \quad x = \frac{192}{3} = 64$$

The number is 64.

- 2) There are 4 small rectangles in this diagram. There are 3 rectangles made up of two smaller ones together, as shown below:



There are 2 rectangles made up of three smaller ones together, as shown below:



And there is the 1 outer rectangle. Then there are  $4 + 3 + 2 + 1 = 10$  rectangles in all.

- 3) Let  $x$  be the third number. Since the average of these numbers is 2,

$$\frac{\frac{5}{3} + \frac{7}{4} + x}{3} = 2; \quad \frac{5}{3} + \frac{7}{4} + x = 6; \quad \frac{20}{12} + \frac{21}{12} + x = 6; \quad \frac{41}{12} + x = \frac{72}{12}; \quad x = \frac{31}{12}.$$

- 4) First,  $\angle BAC = \angle BCA$  because  $\triangle ABC$  is isosceles. Since the sum of the angles in any triangle is  $180^\circ$ ,

$$\angle BAC + \angle BCA + 40^\circ = 180^\circ$$

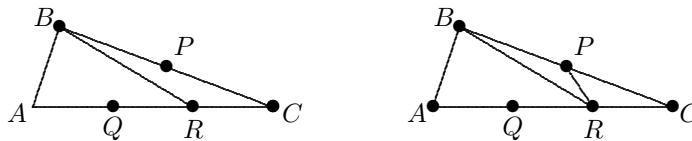
$$2\angle BCA + 40^\circ = 180^\circ$$

$$2\angle BCA = 140^\circ$$

$$\angle BCA = 70^\circ$$

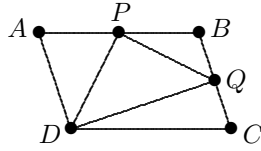
Since the lines  $AC$  and  $BD$  are parallel,  $\angle CBD = \angle BCA$ . Therefore,  $\angle CBD = 70^\circ$ .

- 5) There are  $5 \times 3 = 15$  children in the community. Since each child has 6 toys, there are  $15 \times 6 = 90$  toys in the community.
- 6) The differences between terms are 1, 3, 5, 7, 9,  $\dots$ . In this case, the next difference will be 11. Then the next term is  $28 + 11 = 39$ .
- 7) According to the first step in this multiplication,  $A \times A$  is a number ending in 9. The only possibilities are  $A = 3$  and  $A = 7$ . If  $A = 3$  then the product is  $3 \times 33 = 99$ . But this is not a three-digit number. Then the product must be  $7 \times 77 = 539$ . Therefore,  $B = 5$ .
- 8) Since  $Q$  and  $R$  divide  $AC$  into three equal parts, the area of  $\triangle BRC$  is one-third of the area of  $\triangle BAC$ . Then the area of  $\triangle BRC$  is 12. Since  $P$  divides  $BC$  into two equal parts, the area of  $\triangle PRC$  is half the area of  $\triangle BRC$ . Then the area of  $\triangle PRC$  is 6.



- 9) The last digit (in fact, the only digit) of  $7^1$  is 7; the last digit of  $7^2$  is 9; the last digit of  $7^3$  is 3; and the last digit of  $7^4$  is 1. Continuing, the last digits of  $7^5, 7^6, 7^7, 7^8, 7^9, \dots$ , are 7, 9, 3, 1, 7,  $\dots$ . In fact, the last digits of the powers of 7 form the sequence 7, 9, 3, 1, 7, 9, 3, 1, 7, 9,  $\dots$ . This repeats every 4 terms and continues forever. Note that the 4th, 8th, 12th, and so on, terms are all 1, and this means that the 2008th term is 1. Then the 2009th term is 7 and the 2010th term is 9. Therefore, the last digit of  $7^{2010}$  is 9.
- 10) There will be 12 straws at the base of the grid, 11 straws in the row above it, 10 straws in the row above that one, and so on. Then the total number of straws on the base or parallel to it is  $12 + 11 + 10 + \dots + 1 = 78$ . There are also straws forming the sides of the triangles. On one side, there will be 12 straws on the boundary. Parallel to them, there will be lines with 11 straws, 10 straws, 9 straws, and so on, down to one straw. Then there will be  $12 + 11 + 10 + \dots + 1 = 78$  straws on this boundary or in a parallel direction. Finally, there will also be a total of 78 straws on the third boundary or in lines parallel to it. Then there will be a total of  $78 + 78 + 78 = 234$  straws making the grid.

- 11) First, the area of  $\triangle DAB$  is half of the area of  $ABCD$ . Then the area of  $\triangle DAB$  is 12.



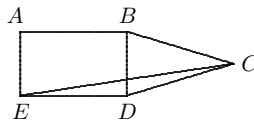
Also, since  $P$  is the midpoint of  $AB$ , the area of  $\triangle DAP$  is half the area of  $\triangle DAB$ . Then the area of  $\triangle DAP$  is 6. By reasoning in the same way, one sees that the area of  $\triangle DCQ$  is 6 as well. Similarly, the area of  $\triangle PQB$  is half the area of  $\triangle PCB$ , and this is half the area of  $\triangle ACB$ , and this is half the area of  $ABCD$ . Then the area of  $\triangle PQB$  is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 24 = 3.$$

Finally, the area of  $\triangle DPQ$  is the area of  $ABCD$  minus the areas of  $\triangle DAP$ ,  $\triangle DCQ$ , and  $\triangle PQB$ . That is, the area of  $\triangle DPQ$  is  $24 - (6 + 6 + 3) = 24 - 15 = 9$ .

- 12) The first letter can be any one from A to Y. (The first letter cannot be Z because there is no letter after Z in the alphabet.) If the first letter is A, the second letter can be any of B, C, D,  $\dots$  Z. There are 25 possibilities in all. If the first letter is B, the second letter can be any of C, D, E,  $\dots$  Z. There are 24 possibilities in all. If the third letter is C, there are 23 possibilities for the second letter. Continuing in this way, there are 22 possibilities after a D, 21 possibilities after an E, and so on, up to one possibility for Y. The total number of pairs of letters is  $25 + 24 + 23 + \dots + 2 + 1 = 325$ . (Note: there is a “fast formula” for adding the numbers  $1 + 2 + 3 + \dots + n$ . This states that the sum is  $n(n+1)/2$ . In our case,  $1 + 2 + 3 + \dots + 25 = 25(25+1)/2 = 25(13) = 325$ .)

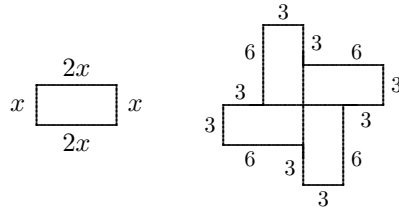
- 13) Since  $BCD$  is an isosceles triangle,  $\angle CBD = \angle CDB$ . Also the sum of three angles in any triangle is  $180^\circ$ . Since  $\angle BCD = 40^\circ$  it follows that  $\angle CBD = \angle CDB = 70^\circ$ .



Since  $\angle BDE = 90^\circ$  we also have  $\angle EDC = 160^\circ$ . Since the triangle  $EDC$  is isosceles,  $\angle DEC = \angle DCE$ . It follows that  $\angle CED = 10^\circ$ . Since  $\angle AED = 90^\circ$  we must have  $\angle AEC = 80^\circ$ .

- 14) The prime factorization of 2010 is  $2 \times 3 \times 5 \times 67$ . To produce a divisor for 2010, we have two choices for the number of 2's which may be used (none or one), two choices for the number of 3s (none or one), two choices for the number of 5s (none or one), and two choices for the number of 67's (none or one). There are a total of  $2 \times 2 \times 2 \times 2 = 16$  possible divisors. They are: 1, 2, 3, 5, 6, 10, 15, 30, 67, 134, 201, 335, 402, 670, 1005, and 2010.

- 15) First,  $\triangle ABC$  and  $\triangle BDC$  are right triangles and both have the angle at  $C$  in common. Since the sum of the angles in any triangle is  $180^\circ$ , it follows that  $\angle DBC = \angle BAC$ . Then  $\triangle ABC$  and  $\triangle BDC$  are similar. Since corresponding sides of similar triangles are proportional,  $BC/AC = CD/BC$ . Since  $BC = 3$  and  $AC = 5$ , we have  $3/5 = CD/3$ . By cross-multiplying, we have  $5CD = 9$  and so  $CD = 9/5$ .
- 16) Let  $ab$  be a two-digit number, where  $a$  and  $b$  are each one of the numbers  $0, 1, 2, \dots, 9$ . Then the value of  $ab$  is  $10a + b$ . When the digits are reversed we obtain the number  $ba$ , with a value of  $10b + a$ . In order to be reflexive, we must have  $(10a + b) - (10b + a) = 27$ . This means that  $10a + b - 10b - a = 27$  and so  $9a - 9b = 27$ . In turn, this means that  $9(a - b) = 27$  and so  $a - b = 3$ . The possibilities for  $a$  and  $b$ , respectively, are  $3$  and  $0$ ,  $4$  and  $1$ ,  $5$  and  $2$ ,  $6$  and  $3$ ,  $7$  and  $4$ ,  $8$  and  $5$ , and  $9$  and  $6$ . That is, the reflexive numbers are  $30, 41, 52, 63, 74, 85$ , and  $96$ . There are  $7$  of them in all.
- 17) Let  $x$  be the width of a single rectangle and  $2x$  be its length. Then its perimeter is  $2x + x + 2x + x = 6x$ . Equating this to  $18$  gives  $x = 3$ . Then each small rectangle has



width  $3$  and length  $6$ . It follows that the lengths of the segments in the boundary of the figure are as shown. The perimeter of the figure is  $48$

- 18) First, since  $a^2 + b^2 = 4$  we have  $a^2 = 4 - b^2$ . Then

$$2a^2 - 3b^2 = 2(4 - b^2) - 3b^2 = 8 - 2b^2 - 3b^2 = 8 - 5b^2.$$

Also,  $b^2 \geq 0$  because all perfect squares are positive or  $0$ . Then  $5b^2 \geq 0$  and it follows that  $8 - 5b^2 \leq 8$ . On the other hand, if  $a = 2$  and  $b = 0$  then  $a^2 + b^2 = 2^2 + 0^2 = 4$ . Also,  $2a^2 - 3b^2 = 2(2^2) - 3(0^2) = 8 - 0 = 8$ . Then the greatest possible value for  $2a^2 - 3b^2$  is  $8$ .

- 19) Since  $A + 1 = B - 2$  we have  $B = A + 3$ . Since  $A + 1 = C + 3$  we have  $C = A - 2$ . Since  $A + 1 = D - 4$  we have  $D = A + 5$ . Finally, since  $A + 1 = E + 5$  we have  $E = A - 4$ . Then

$$\begin{aligned} A + B + C + D + E &= A + (A + 3) + (A - 2) + (A + 5) + (A - 4) \\ &= 5A + (3 - 2 + 5 - 4) \\ &= 5A + 2 \end{aligned}$$

- 20 The first few three-digit numbers which are odd and divisible by 3 are 105, 111, 117, ... In general, if  $n$  is odd and divisible by 3 then the next such number is  $n + 6$ . Since 105 is the first odd number divisible by 3, the others we seek are of the form  $105 + 6k$  for some  $k$ . For these to be less than 456 we must have  $105 + 6k \leq 456$  and thus  $6k \leq 351$ . Since  $k$  is a natural number, the solution is  $k \leq 58$ . Starting with 105 (when  $k = 0$ ) and ending with 453 (when  $k = 58$ ), there are 59 such numbers in total.
- 21) Since  $\triangle ABC$  is a right triangle,  $(AC)^2 + (BC)^2 = (AB)^2$  by the Pythagorean theorem. Since  $AC = 12$  and  $AB = 20$ ,

$$12^2 + (BC)^2 = 20^2; \quad 144 + (BC)^2 = 400; \quad (BC)^2 = 256; \quad BC = 16.$$

Also, the area of  $\triangle ABC$  is  $\frac{1}{2} \times \text{height} \times \text{width} = \frac{1}{2} \times 16 \times 12 = 96$ . Second,  $\triangle EBD$  is a right triangle and  $\triangle ABC$  and  $\triangle EBD$  have the angle at  $B$  in common. Since the sum of the angles in any triangle is  $180^\circ$ , it follows that  $\angle BED = \angle BAC$ . Then  $\triangle ABC$  and  $\triangle EBD$  are similar. Since corresponding sides of similar triangles are proportional,  $DE/BD = AC/BC$ . This means that

$$\frac{DE}{10} = \frac{12}{16} = \frac{3}{4}; \quad 4DE = 30; \quad DE = \frac{30}{4} = \frac{15}{2}$$

Since the area of  $\triangle BDE$  is  $\frac{1}{2} \times \text{height} \times \text{width}$ , its area is  $\frac{1}{2}(10)(\frac{15}{2}) = \frac{75}{2}$ . Finally, if we subtract the area of  $\triangle BDE$  from that of  $\triangle ABC$  we obtain the area of  $ADEC$ . This area is  $96 - \frac{75}{2} = \frac{192}{2} - \frac{75}{2} = \frac{117}{2}$ .

- 22) The perfect squares less than 18 are 1, 4, 9, and 16. If 16 is the largest summand then 18 can be written in one way:  $16 + 1 + 1$ . If 9 is the largest summand then 18 can be written in four ways:  $9 + 9$ ,  $9 + 4 + 4 + 1$ ,  $9 + 4 + 1 + 1 + 1 + 1 + 1$ , and  $9 + 1 + 1 + \dots + 1$ . If 4 is the largest summand then 18 can be written in four ways:  $4 + 4 + 4 + 4 + 1 + 1$ ,  $4 + 4 + 4 + 1 + 1 + 1 + 1 + 1 + 1$ ,  $4 + 4 + 1 + 1 + 1 + \dots + 1$ ,  $4 + 1 + 1 + 1 + \dots + 1$ . If 1 is the largest summand then 18 can be written in exactly one way (using eighteen 1s). In total there are 10 ways of writing 18 as a sum of perfect squares.
- 23) The least common multiple of 4, 5, and 6 is 60. So in order for a number to be divisible by 4, 5, and 6 it must be a 60-multiple. The 60-multiples less than 1000 are 60, 120, 180, ..., 960. There are 16 such numbers in total.
- 24) The figures have 2, 3, 4, ... rows and 3, 5, 7, ... columns. That is, the  $n$ th figure has  $n + 1$  rows and  $2n + 1$  columns. When  $n = 8$  the figure has 9 rows and 17 columns, for a total of 153 stars.

25) To say that 35 is a divisor of a number  $n$  is to say that  $n$  is a 35-multiple. Then we can write  $n = 35m$  for some natural number  $m$ . The cases with  $m = 1, 2, 3, 4,$  and  $5$  are indicated in the table below.

<b>m</b>	<b>n</b>	<b>Largest Divisor</b>
1	35	7
2	70	35
3	105	35
4	140	70
5	175	35

If  $m = 2, 3,$  or  $5$  then 35 is the largest divisor of  $n$ . If  $m$  is 6 or more then  $n = 5(7)m$ . In this case  $7m$  is a factor of  $n$  and  $7m > 35$  (because  $m > 5$ ). Then the only possibility that 35 is the largest factor of  $n$ , other than  $n$  itself, is when  $n = 70, 105,$  or  $175$ . There are three such numbers.