THE 2010 JAMAICAN MATHEMATICAL OLYMPIAD

Presented by The University of the West Indies
In Collaboration with Sterling Asset Management Ltd

Qualifying Round
Solutions for Grades 9, 10, and 11

1) Let x be the number. Then

$$\frac{4}{5}x = 60; \qquad \frac{4x}{5} = \frac{60}{1}; \qquad 4x = 300; \qquad x = 75$$

The number is 75.

2) There are 12 small triangles in this figure. In addition there are 4 larger triangles, with each one made up of four smaller triangles as shown below.

There are 16 triangles in all.

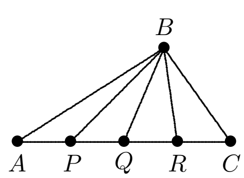
3) This sequence repeats every 7 terms. Then the 7th, 14th, 21st, 28th, 35th, and so on, terms are all *ti*. This continues up to the 2009th term, which is also *ti*. Then the 2010th term is *doh*.

4) Let c be the number of chickens and r the number of rabbits on the farm. Then $c + r = 25$. Since each chicken has 2 legs and each rabbit has 4 legs, $2c + 4r = 84$. Multiplying the first equation by 2 gives the system

$$\begin{cases} 2c + 2r = 50 \\ 2c + 4r = 84 \end{cases}$$

Subtracting the first equation from the second, $2r = 34$ and so $r = 17$. Then $c + 17 = 25$ and so $c = 8$. There are 8 chickens on the farm.

5) Since P, Q, and R divide AC into 4 equal parts, $\triangle BAP$, $\triangle BPQ$, $\triangle BQR$, and $\triangle BRC$ have



equal areas. We will write $a(BAP) = a(BPQ) = a(BQR) = a(BRC)$ in this case. (So, $a(BAP)$ stands for the area of $\triangle BAP$, and so on.) Then

$$a(BQR) + a(BRC) = a(BQC); \qquad 2 \times a(BQR) = 24; \qquad a(BQR) = 12.$$

Then we have $a(BAR) = a(BAP) + a(BPQ) + a(BQR) = 12 + 12 + 12 = 36$.

6) The last digit (in fact, the only digit) of 3^1 is 3; the last digit of 3^2 is 9; the last digit of 3^3 is 7; and the last digit of 3^4 is 1. Continuing, the last digits of 3^5, 3^6, 3^7, 3^8, 3^9, ..., are 3, 9, 7, 1, 3, ... In fact, the last digits of the powers of 3 form the sequence 3, 9, 7, 1, 3, 9, 7, 1, 3, 9, ... This repeats every 4 terms and continues forever. Note that the 4th, 8th, 12th, and so on, terms are all 1, and this means that eventually the 2008th term is 1. Then the 2009th term is 3 and the 2010th term is 9. Therefore, the last digit of 3^{2010} is 9.

7) By trial and eror we see that $58^2 = 3,364$ and $59^2 = 3,481$. Then $58^2 < 3,456 < 59^2$. It follows that $N = 58$.

8) The number of men in the town is $\frac{1}{2} \times 1,200 = 600$. The number of men who play dominoes is $\frac{1}{3} \times 600 = 200$. The number of male dominoe players who enjoy golf is $\frac{1}{4} \times 200 = 50$. There are 50 men who play dominoes and enjoy golf.

9) The sum of the angles in any triangle is 180°. Since $\angle BAC = \angle DAC = 25°$,

$$25° + 90° + \angle CBA = 180°; \qquad 115° + \angle CBA = 180°; \qquad \angle CBA = 65°$$

This means that $\angle CBD = 65°$ as well. Since $\triangle DBC$ is isosceles, we also have $\angle BCD = 65°$. Furthermore, the sum of the angles in any triangle is 180°. Then

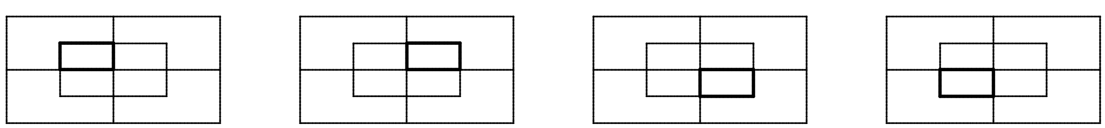
$$\angle BDC + 65° + 65° = 180°; \qquad \angle BDC + 130° = 180°; \qquad \angle BDC = 50°$$

10) If a number is divisible by 3 and 11 then it is divisible by 33. In other words, it is a multiple of 33. The smallest multiple of 33 which has three digits is 132. This is 4×33. The next multiples of 33 are 165, 198, 231, and so on. These are 5×33, 6×33, 7×33, and so on. The highest multiple of 33 which less than 1000 is 990. This is 30×33. Then the number of three-digit numbers divisible by both 3 and 11 is the same as the number of numbers in the set $\{4, 5, 6, 7, \ldots, 30\}$. This is 27.

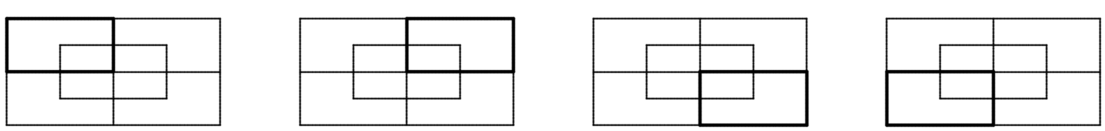
11) If $a^2 + 6b^2 = 7ab$ then $a^2 - 7ab + 6b^2 = 0$. Thus $(a - b)(a - 6b) = 0$. Then, possibly, $a - b = 0$ and so $a = b$. Otherwise, $a - 6b = 0$ and so $a = 6b$. Since a and b re positve and $a > b$, we must have $a = 6b$. Then

$$\frac{a}{b} = \frac{6b}{b} = 6$$

12) First, there are 4 small rectangles as illustrated below. Also, these rectangles can be paired



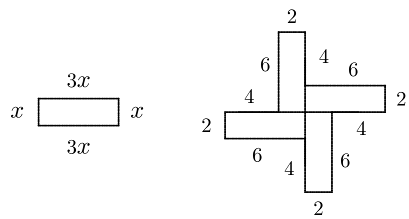
to make larger rectangles in 4 different ways. Finally, the four small rectangles together make a single larger rectangle. Next, there are 4 larger rectangles as shown below. Also, these rectangles



can be paired to make larger rectangles in 4 different ways. Finally, there is also the outermost rectangle. A total of $4 + 4 + 1 + 4 + 4 + 1 = 18$ rectangles may be found in the diagram.

13) In its decimal form, the number 10^{2010} consists of a 1 followed by a 0 written 2010 times. Subtracting 2010 from this gives the number represented by a 9 written 2006 times followed by 7990. The sum of the digits in this number is $2006(9) + 7 + 9 + 9 + 0 = 18,079$.

14) Let x be the width of a rectangle and $3x$ be its length. Since the area of each rectangle is $12\,\text{cm}^2$, we have $3x^2 = 12$. Then $x^2 = 4$ and so $x = 2$. It follows that segments in the figure have the lengths shown. The perimeter of the figure is 48.



15) Since $(999,999,999,876)^2 - (124)^2$ is the difference of two squares, it may be factored as

$$(999,999,999,876 + 124)(999,999,999,876 - 124) = (1,000,000,000,000)(999,999,999,752)$$

The number 999,999,999,752 obviously has 12 digits. If it is multiplied by 1,000,000,000,000 then the effect will be to add 12 zeroes to the end of this number. Therefore, the total number of digits in $(999,999,999,876)^2 - (124)^2$ is 24.

16) Since the only divisor of 1 is 1, it does not have exactly 4 divisors. Suppose now that n is a natural number and consider its prime factorization. If this factorization contains three distinct primes, then n has more than 4 divisors. To see this, suppose p, q, and r are prime factors of n. Then 1, p, q, r, and pqr are all divisors of n. So, if n has exactly 4 divisors it must have only 1 or 2 primes in its factorization. Suppose n has two primes, p and q, in its factorization. Let p represent the smaller one and q the larger one. Then 1, p, q, and pq are factors of n. If n has exactly 4 factors then these are the only ones. Then $pq = n$. In this case, it is possible that $p = 2$, 3, 5, or 7. (Otherwise, if $p \geq 11$ then $q \leq 75/11$. This means that $q \leq 7$ and so $q < p$.) If $p = 2$ then q could be 3, 5, 7, 11, 13, 17, 23, 29, 31, or 37. There are 10 choices in this case. If $p = 3$ then q could be 5, 7, 11, 13, 17, or 23. There are 6 choices in this case. If $p = 5$ then q could be 7, 11, or 13. There are 3 choices in this case. If $p = 7$ then there are no possible choices available. (The next prime is 11 and 7×11 is too large.) Then the total number of choices for primes p and q is $10 + 6 + 3 = 19$.

Suppose now that n has only one prime, p, in its factorization. In order to have exactly 4 divisors they would have to be 1, p, p^2, and p^3. Also, we would have $p^3 = n$ in this case. The possiblities are $p = 2$, $n = 8$, or $p = 3$, $n = 27$, or $p = 4$, $n = 64$. There are 3 choices in this case. Adding these to the case we analyzed earlier, there are 22 numbers less than 75 which have exactly 4 divisors.

17) By the Pythagorean theorem, $(AB)^2 + (BC)^2 = (AC)^2$. Substituting $AC = 13$ and $AB = 12$,

$$12^2 + (BC)^2 = 13^2; \qquad 144 + (BC)^2 = 169; \qquad (BC)^2 = 25; \qquad BC = 5.$$

Also $\triangle BDC$ is a right triangle and $\triangle ABC$ and $\triangle BDC$ have a common angle at C. Since the sum of the angles of any triangle is $180°$, it follows that $\angle DBC = \angle BAC$. In this case, the corresponding angles in $\triangle ABC$ and $\triangle BDC$ are equal. Therefore, these triangles are similar. Then

$$\frac{BD}{BC} = \frac{AB}{AC}; \qquad \frac{BD}{5} = \frac{12}{13}; \qquad 13BD = 60; \qquad BD = \frac{60}{13}$$

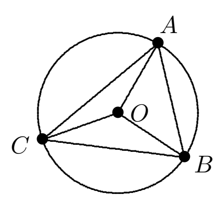
18) The equation $x^2 + ax + b$ has exactly one real root when $a^2 - 4b = 0$. The values for a and b that satisfy this relation are $a = 2$, $b = 1$; $a = 4$, $b = 4$; $a = 6$, $b = 9$; $a = 8$, $b = 16$; $a = 10$, $b = 25$; $a = 12$, $b = 36$. Any other choices for a and b will have the property that $a + b > 50$. Then we can choose a and b in exactly 6 ways.

19) Let $a_1 = m$ and $a_2 = n$. Then

$$\begin{aligned} a_3 &= a_1 + a_2 = m + n \\ a_4 &= a_2 + a_3 = m + 2n \\ a_5 &= a_3 + a_4 = 2m + 3n \\ a_6 &= a_4 + a_5 = 3m + 5n \end{aligned}$$

Therefore, m and n are natural numbers such that $m < n$ and $3m + 5n = 46$. In this case, $5n = 46 - 3m$ and so $46 - 3m$ must be a 5-multiple. The only m-values for which $46 - 3m$ is a 5-multiple are $m = 2$, $m = 7$, and $m = 12$. Then there are three candidates for m and n: $m = 2$ and $n = 8$; $m = 7$ and $n = 5$; $m = 12$ and $n = 2$. Since we must also have $m < n$, the only possible solution is $m = 2$ and $n = 8$. In this case, $a_3 = m + 2n = 2 + 2(8) = 2 + 16 = 18$.

20) Since OA, OB, and OC are radial segments, they are equal in length. Then $\triangle AOB$, $\triangle BOC$,



and $\triangle COA$ are isosceles triangles. This means that $\angle OAB = \angle OBA$, $\angle OBC = \angle OCB$, and $\angle OCA = \angle OAC$. Also, the sum of the angles in $\triangle ABC$ is $180°$. Then

$$\begin{aligned} \angle OAB + \angle OBA + \angle OBC + \angle OCB + \angle OCA + \angle OAC &= 180° \\ 2\angle OAB + 2\angle OBC + 2\angle OCA &= 180° \\ \angle OAB + \angle OBC + \angle OCA &= 90° \\ 40° + 30° + \angle OCA &= 90° \\ 70° + \angle OCA &= 90° \\ \angle OCA &= 20° \end{aligned}$$

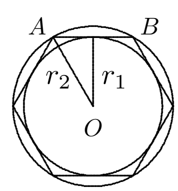
21) By the binomial formula,

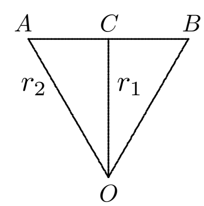
$$5^{2010} = (6-1)^{2010} = 6^{2010} + c_1 6^{2009}(-1) + c_2 6^{2008}(-1)^2 + c_3 6^{2007}(-1)^3 + \cdots$$
$$+ c_{2008} 6^2 (-1)^{2008} + c_{2009} 6(-1)^{2009} + (-1)^{2010}$$

Here, each c_k represents the kth binomial coefficient $2010!/\big(k!(2010-k)!\big)$. Also, each term except the last one, $(-1)^{2010}$, is a multiple of 6. Then each term except the last one is divisible by 3. Then the remainder when 5^{2010} is divided by 3 will be $(-1)^{2010} = 1$.

22) We must consider three possibilities: two apartments receive two samples each; one apartment receives two samples and two receive one each; or four apartments receive one sample each. In the first case, there are 10 ways of choosing the two apartments to receive the samples: AB, AC, AD, AE, BC, BD, BE, CD, CE, or DE. In the second case, there are 5 ways to choose one apartment to receive two samples. For each such choice, there are six ways to choose two more apartments to receive one sample each. Then there are a total of $5 \times 6 = 30$ ways to distribute the samples in this case. In the third case, there are 5 ways to choose the four apartments to receive the samples: ABCD, ABCE, ABDE, ACDE, or BCDE. Then there are $10 + 30 + 5 = 45$ ways of delivering the samples.

23) Let r_1 be the radius of the inner circle and r_2 the radius of the outer circle. Then $A_1 = \pi r_1^2$





and $A_2 = \pi r_2^2$. Also, as seen in the enlarged diagram, $\angle AOB$ is $60°$. This is because this angle is one-sixth of a full circle and so $\angle AOB = 360°/6 = 60°$. Therefore, $\angle AOC = 30°$. Then

$$\frac{r_1}{r_2} = \cos 30° = \frac{\sqrt{3}}{2}; \qquad r_1 = \frac{\sqrt{3}\, r_2}{2}; \qquad r_1^2 = \frac{3r_2^2}{4}$$

Finally,

$$\frac{A_2}{A_1} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{\pi\left(\dfrac{3r_2^2}{4}\right)}{\pi r_2^2} = \frac{3\pi r_2^2}{4\pi r_2^2} = \frac{3}{4}.$$

24) By trial and error, one may verify that $1+2+3+\cdots+54 = 1,485$ and $1+2+3+\cdots+54+55 = 1540$. Then $N = 54$. Alternatively, there is a "fast formula" for finding a sum of the form $1+2+3+\cdots+n$. Let S represent this sum. Then

$$\begin{cases} S = 1 + 2 + 3 + \cdots + n \\ S = n + (n-1) + (n-2) + \cdots + 1 \end{cases}$$

Adding equations, $2S = (n+1)+(n+1)+(n+1)+\cdots+(n+1)$, where the term $n+1$ occurs on the right side n times. Then $2S = n(n+1)$ and so $S = n(n+1)/2$. In this problem, we still need some amount of trial and error. However, one verifies that $1+2+3+\cdots+54 = 54(54+1)/2 = 54(55)/2 = 1485$. Similarly, $1+2+3+\cdots+55 = 55(55+1)/2 = 55(56)/2 = 1540$.

25) Let ℓ_1 and ℓ_2 be the side lengths of S_1 and S_2, respectively. Then $A_1 = \ell_1^2$ and $A_2 = \ell_2^2$. Also, $P_1 = 4\ell_1$ and $P_2 = 4\ell_2$. We have

$$\frac{A_1}{A_2} = 5; \qquad \frac{\ell_1^2}{\ell_2^2} = 5; \qquad \ell_1^2 = 5\ell_2^2; \qquad \ell_1 = \sqrt{5}\ell_2.$$

(We choose the positive square root above because the length ℓ_1 cannot be negative.) Then

$$\frac{P_1}{P_2} = \frac{4\ell_1}{4\ell_2} = \frac{\ell_1}{\ell_2} = \frac{\sqrt{5}\ell_2}{\ell_2} = \sqrt{5}.$$

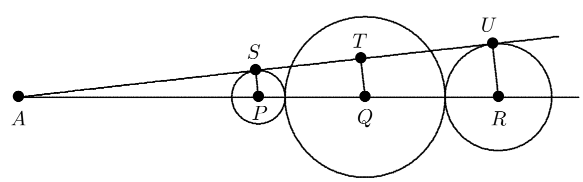
26) The first step in this multiplication indicates that $D \times AB = AB$. Then $D = 1$. Also, adding the two partial answers indicates that $A + 5$ gives an answer ending in 2. Since A is a digit from 0 to 9, the only possibility is $A = 7$. Then $7 + 5 = 12$. After the 2 is recorded the tens digit, 1, is carried to the hundreds place. Then $1 + E = 7$ and so $E = 6$. To complete the analysis, it is clear from the multiplication of the tens digits that $B = 3$.

27) Since squares of real numbers are always positive or 0, we have $b^2 \geq 0$. Similarly, $a^2 \geq 0$. Since $a^2 = 10 - b^2$ we have $10 - b^2 \geq 0$ and so $10 \geq b^2$. That is, $b^2 \leq 10$. Thus $0 \leq b^2 \leq 10$. Since $a^2 + b^2 = 10$,

$$2a^2 + 3b^2 = 2a^2 + 2b^2 + b^2 = 2(a^2 + b^2) + b^2 = 2(10) + b^2 = 20 + b^2.$$

Then $2a^2 + 3b^2 = 20 + b^2 \leq 30$. Furthermore, $2a^2 + 3b^2 = 30$ when $a = 0$ and $b = \pm\sqrt{10}$. Then the largest possible value for $2a^2 + 3b^2$ is 30.

28) Let A be the point where the lines SU and PR meet. Since AU is tangent to the first and



third circles, $\angle ASP$ and $\angle AUR$ are right angles. Then$\triangle ASP$, $\triangle ATQ$, and $\triangle AUR$ are right triangles. Also, they all have the same angle at A. Since the sum of the angles in any triangle is 180°, it follows that $\angle APS$, $\angle AQT$, and $\angle ARU$ are equal. Therefore, the triangles APS, AQT, and ARU are similar. Since similar triangles have proportional sides,

$$\frac{AP}{SP} = \frac{AR}{UR}$$

The circles at P, Q, and R have radii 2, 6, and 4, respectively. Then $AR = AP + 2 + 6 + 6 + 4 = AP + 18$. By substitution,

$$\frac{AP}{2} = \frac{AP + 18}{4}; \qquad 4AP = 2AP + 36; \qquad 2AP = 36; \qquad AP = 18.$$

To solve for TQ, note that $AQ = AP + 2 + 6 = 18 + 2 + 6 = 26$. Then

$$\frac{TQ}{AQ} = \frac{SP}{AP}; \qquad \frac{TQ}{26} = \frac{2}{18}; \qquad \frac{TQ}{26} = \frac{1}{9}; \qquad 9TQ = 26; \qquad TQ = \frac{26}{9}$$

29) Note first that m may be any of the numbers 1, 2, 3, ..., 18. (We cannot have $m = 19$ because $19 + 1$ is not less than 20.) If $m = 1$ then n may be any one of 1, 2, 3, ..., 18. There are 18 choices in all. If $m = 2$ then n may be any one of 1, 2, 3, ..., 17. There are 17 choices in all. If $m = 3$ then n may be any one of 1, 2, 3, ..., 16. There are 16 choices in all. In this way, when $m = 4, 5, 6, \ldots, 18$, there are 14, 13, 12, ..., 1 choices for n. Then the total number of pairs (m, n) is $1 + 2 + 3 + \cdots + 18 = 171$.

30) Suppose the number is $d_1d_2d_3d_4d_5d_6$. Then the digits must be 1, 2, 3, 4, 5, and 6 in some order. Also, we must have d_1d_2 divisible by 2, $d_1d_2d_3d_4$ divisible by 4, and $d_1d_2d_3d_4d_5d_6$ divisible by 6. This means that d_2, d_4, and d_6 must be even. Then the number has the form

$$O\ E\ O\ E\ O\ E,$$

Where O represents an odd digit and E represents an even digit. Also, $d_1d_2d_3d_4d_5$ must be divisible by 5. This means that its last digit must be 0 or 5. Since 0 is not possible, $d_5 = 5$. Then the number looks like

$$O\ E\ O\ E\ 5\ E.$$

Note that the first and third digits are 1 and 3 in some order. If $d_2 = 4$ then $d_1d_2d_3$ must be 143 or 341. However, neither is divisible by 3. If $d_2 = 6$ then $d_1d_2d_3$ must be 163 or 361. However, neither is divisible by 3. Therefore $d_2 = 2$, and the number has the form

$$O\ 2\ O\ E\ 5\ E.$$

If $d_4 = 4$ then $d_1d_2d_3d_4$ is either 1234 or 3214. However, neither one is divisible by 4. Therefore, $d_4 = 6$. By process of elimination, $d_6 = 4$. Then the number has the form

$$O\ 2\ O\ 6\ 5\ 4.$$

There are two possibilities remaining: 123,654 or 321,654. In fact, both are curious. Then there are exactly 2 six-digit curious numbers.