The University of the West Indies, Mona

presents

The 2022 Senior Mathematics Olympiad

Qualifying Round Examination (SOLUTIONS)

1. **Soln (D)** Let x be the number of short women. The number of tall women is therefore $31-x$. The number of men is $50-31=19$ and so the number of short men is 5. Since the number of short persons is 18, we have $5+x=18$. Solving, $x=13$.

2. **Soln (C)** Let the numbers of red and blue marbles be r and b. $\dfrac{3}{5}=\dfrac{b}{b+r}$. This gives $5b=3b+3r$ or $2b=3r$ or $b=\frac{3}{2}r$. If the number of red marbles is doubled and the number of blue marbles stays the same, the total is $2r+b$ and the fraction of red marbles is
$$\frac{2r}{2r+b}=\frac{2r}{2r+\frac{3}{2}r}=\frac{4r}{4r+3r}=\frac{4}{7}$$

3. **Soln (B)** Suppose Arlene is responsible and is lying. This would mean that Barbara is also lying and so Arlene is not the person. Suppose Barbara is responsible and is lying. In this case, Arlene, Canute and Daniel's statements are consistent. Suppose Canute is responsible and is lying. This would mean that Arlene is also lying and so Canute is not the person. Similarly, Duncan cannot be the persons. The correct answer is B.

4. **Soln (C)** Let x be the number of correct response, y the number of incorrect and z the number of questions unanswered. His score is $5x-2y=48$. where $x+y+z=20$. So
$$\begin{aligned} 5x-2y &= 48 \\ 2x+2y+2z &= 40 \end{aligned}$$
Adding, $7x+2z=88$ or $7x=88-2z$. Now x is greatest when z is least. The least such z is 2 corresponding to $x=12$

5. **Soln (D)** Clearly the largest amount is received when only one person gets more than \$5 and when the other 7 persons receive as small as possible. Let m be the largest amount. We therefore want
$$m+2+2+2+2+1+1+1=30$$
This corresponds to $m=19$.

6. **Soln (A)** Suppose the original amount of money is \$$x$. Cecil receives

$$\begin{aligned}
x - \left[\left(1+\frac{2}{3}(x-1)\right)+6+\frac{1}{3}\left(x-\left(1+\frac{2}{3}(x-1)\right)-6\right)\right] &= 40 \\
x-1-\frac{2}{3}x+\frac{2}{3}-6-\frac{1}{3}x+\frac{1}{3}+\frac{2}{9}x-\frac{2}{9}+2 &= 40 \\
\frac{2}{9}x-\frac{38}{9} &= 40 \\
2x-38 &= 360 \\
2x &= 398 \\
x &= 199
\end{aligned}$$

Byron's share is

$$\begin{aligned}
6+\frac{1}{3}\left(x-\left(1+\frac{2}{3}(x-1)\right)-6\right) &= 6+\frac{1}{3}\left(199-\left(1+\frac{2}{3}(198)\right)-6\right) \\
&= 26
\end{aligned}$$

Alternatively, After Alyah had received his share, Byron received \$6 plus one-third of the remainder. Since Cecil gets the rest, she received two-thirds of the remainder, which is \$40. Thus, one-third of the remainder is \$20. Therefore Byron receives is \$20 + \$6 = \$26.

7. **Soln (B)** Let $x = 20p, y = 20q$ and $z = 20r$ so that

$$20p + 20q + 20r = 180 \text{ or } p + q + r = 9$$

and $p \leq q \leq r \in \mathbb{Z}^+$. The possible triplets are

$$(1,1,7),(1,2,6),(1,3,5),(1,4,4),(2,2,5),(2,3,4),(3,3,3)$$

There are 7 triplets

8. **Soln (E)** Since $3+8+6+9=26$, $9+D+5+1=26$ and so $D=11$. With this information,

$$A+B=22, A+C=14, C+E=9 \text{ and } B+E=17$$

The list of numbers not yet used is $\{2,7,10,12\}$. We therefore have two possibilities for C and E. **Case 1:** $C=7$ and $E=2$. In this case B would be 15 which is not in the list. **Case 2:** $C=2$ and $E=7$. In this case, $B=10$ and $A=12$.

9. **Soln (D)** It is clear that the number of females is 99 and the number of males is 1. This male lives on campus (otherwise all the students living on campus) would be male. The 2% males living on campus is accounted for by this 1 male. The 100% living on campus would therefore be $50 \times 1 = 50$ (49 females and 1 male)There are 7 triplets

10. **Soln (E)** When $a > 0$ and $a = b = c$, the number takes the form $aaa = a\,(111) = a \cdot 3 \cdot 37$. So 37 is a factor of aaa.

11. **Soln (D)** Because three car washers can wash 4 cars in 5 hours, 1 car washer will wash $\frac{4}{3}$ cars in 5 hours or $\frac{4}{15}$ cars in 1 hour. The 7 car washers will now work at a combined rate of $\frac{4}{15} \times 7 = \frac{28}{15}$ cars in 1 hour. If h is the number of hours to wash the 18 cars, then
$$\frac{28}{15}h = 18$$
This gives $h = \frac{18 \times 15}{28} = \frac{135}{14} = 9\frac{9}{14}$. Rounding up, we get 10.

12. **Soln (A)** Let d be the number of days in a week and the number of months be m. So $d \cdot d = d^2$ is the number of days in one month. We therefore have
$$md^2 = 1,250$$
Also,
$$m = 2d^2 \text{ of } d^2 = \frac{m}{2}$$
It follows that
$$m \cdot \frac{m}{2} = 1,250$$
This gives $m^2 = 2500$ and therefore $m = 50$.

13. **Soln (B)** The octagon may be decomposed into 4 congruent rectangles and 8 congruent triangles. The shaded area consists of one such rectangle and 2 of the triangles. If x is the area of one of the rectangle and y is the area of one of the triangle, then the fraction is
$$\frac{x + 2y}{4x + 8y} = \frac{x + 2y}{4\,(x + 2y)} = \frac{1}{4}.$$

14. **Soln (E)** We will count by cases.

    Case 1: The first ball is odd: Odd followed by odd.

Number of ways is $5 \cdot 4 = 20$

Case 2: The first ball is even: Even followed by even

Number of ways is $5 \cdot 4 = 20$

The total is $20 + 20 = 40$

15. **Soln (C)** In all the terms except the first 4, $5 \times 2 = 10$ is a factor and therefore their units digit is 0. The units digit for the the terms of n are therefore

$$1 + 2 + 6 + 4 + 0 + 0 + \cdots + 0 + 0$$

This corresponds to a units digit of 3 for n.

16. **Soln (B)** Let w be the width of the rectangle. $4w = 3\ell$ where ℓ is the length.

$$w + w + \ell + \ell = w + w + \frac{4}{3}w + \frac{4}{3}w = \frac{14}{3}w = 56$$

This gives $w = \frac{3 \times 56}{14} = 12$ The length of the diagonal is

$$\begin{aligned} d &= \sqrt{w^2 + \ell^2} = \sqrt{w^2 + \left(\frac{4}{3}w\right)^2} \\ &= \left(\sqrt{1 + \frac{16}{9}}\right) w = \left(\sqrt{\frac{25}{9}}\right) w = \frac{5}{3}w \end{aligned}$$

Since $w = \frac{42}{5}$, the length of the diagonal is $d = \frac{5}{3}w = \frac{5}{3} \cdot 12 = 20$ cm.

17. **Soln (E)** We observe that

$$\begin{aligned} 2^{5555} &= \left(2^5\right)^{1111} = 32^{1111} \\ 3^{3333} &= \left(3^3\right)^{1111} = 27^{1111} \\ \text{and } 6^{2222} &= \left(6^2\right)^{1111} = 36^{1111} \end{aligned}$$

From this, it is evident that $b < a < c$.

18. **Soln (D)** In seconds

$$\begin{aligned} \text{2000 days} &: 2000 \times 24 \times 60 \times 60 = 48 \times 36 \times 10^5 = 1728 \times 10^5 \text{ seconds} \\ \text{2000 hours} &: 2000 \times 60 \times 60 = 72 \times 10^5 = \text{seconds} \\ \text{2000 minutes} &: 2000 \times 60 = 12 \times 10^4 = \text{seconds} \\ \text{2000 seconds} &: \text{2000 seconds} \end{aligned}$$

The sum is equivalent to $172.8 \times 10^6 + 7.2 \times 10^6 + 1.2 \times 10^4 + 2 \times 10^3$. In millions, this is approximately

$$N \approx 172.8 + 7.2 \approx 180$$

19. **Soln (E)** The area of the original triangle is $A_0 = \frac{1}{2}bh$. The area of the new triangle is $\frac{1}{4}bh$. Let x be the reduction in the base.

$$\frac{1}{2}(b-x)(h+m) = \frac{1}{4}bh$$

The problem now is to find an expression for x.

$$2(b-x)(h+m) = bh$$

Therefore

$$b - x = \frac{bh}{2(h+m)}$$

From this,

$$\begin{aligned} x &= b - \frac{bh}{2(h+m)} = \frac{b \cdot 2(h+m)}{2(h+m)} - \frac{bh}{2(h+m)} \\ &= \frac{2bh + 2bm - bh}{2(h+m)} = \frac{2bm + bh}{2(h+m)} = \frac{b(2m+h)}{2(m+h)}. \end{aligned}$$

20. **Soln (C)** The costs for the red marbles from store A is $\$\frac{1}{4}$. The costs for the blue marbles from store A is $\$\frac{1}{3}$. The costs for the red marbles from store B is $\$\frac{1}{4}$. The costs for the blue marbles from store B is $\$\frac{1}{6}$. Buying m red marbles and n blue marbles from stores A and B, Mary spends, in dollars,

$$\frac{m}{4} + \frac{n}{3} + \frac{m}{4} + \frac{n}{6} = 10$$

Multiplying through by 12 :

$$\begin{aligned} 3m + 4n + 3m + 2n &= 120 \\ 6(m+n) &= 120 \end{aligned}$$

From this $m + n = 20$. The total number of marbles bought is $2(m+n) = 40$.

21. **Soln (D)** It is evident that the radius of the triangle is $r = DB$. From Pythagoras theorem,

$$r = \sqrt{AB^2 + BC^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ units}$$

The area of the shaded region is the area of the sector (quarter circle) minus the area of the rectangle. This is

$$\frac{1}{4} \times \pi \times 5^2 - 3 \times 4 = \frac{25\pi}{4} - 12$$

But $76 \leq 25\pi \leq 80$ and so $19 \leq \dfrac{25\pi}{4} \leq 20$ and

$$19 - 12 \leq \frac{25\pi}{4} - 12 \leq 20 - 12$$

The are is therefore between 7 and 8 square units.

22. **Soln (D)** Let the numbers be x and y where $x < y$

$$y - x : x + y : xy \text{ is } 1 : 7 : 18$$

So

$$\frac{y - x}{x + y} = \frac{1}{7}$$

and so (cross multiplying) $7y - 7x = x + y$. This gives $6y = 8x$ or $y = \frac{4}{3}x$. Since $x + y : xy$ is $7 : 18$

$$\frac{x + \frac{4}{3}x}{x \cdot \frac{4}{3}x} = \frac{7}{18} \text{ or } \frac{\frac{7}{3}}{\frac{4}{3}x} = \frac{7}{18} \text{ or } \frac{7}{4x} = \frac{7}{18}$$

This means that $4x = 18$ or $x = \frac{9}{2}$. It follows that the product $xy = \frac{9}{2} \cdot \left(\frac{4}{3} \cdot \frac{9}{2}\right) = 27$.

23. **Soln (E)** At $3:00$ the minute hand is vertical and the hour hand is horizontal.. At $3:26$, let the angle between the vertical (at 12) and the minute hand be a and the angle between the horizontal (at 3) and the hour hand be b.

$$a = \frac{26}{60} \times 360^\circ = 156^\circ$$

This is $180^\circ - 156^\circ = 24^\circ$ from the vertical at 6. Also

$$b = \frac{26}{60} \times \frac{1}{12} \times 360^\circ = 13^\circ$$

This is the angle below the horizontal at 3. The required angle is $90^\circ - 24^\circ - 13^\circ = 53^\circ$.

24. **Soln (D)** From, πr^2, the areas of the two circles are π and 9π. If A is the unshaded area in the small circle and B is the unshaded area of the bigger circle, then

$$A+\frac{\pi}{3}=\pi \text{ and } B+\frac{\pi}{3}=9\pi$$

Adding, $A+B+\dfrac{2\pi}{3}=10\pi$, From this, $A+B=\dfrac{28\pi}{3}$.

25. **Soln (A)** The interior angles of a regular pentagon are $\frac{1}{5}\times 3\times 180^\circ = 108^\circ$. Since angle $PAE = PEA = 60^\circ$, angle

$$PAB = PED = 108^\circ - 60^\circ = 48^\circ$$

Furthermore, because triangles BAP and DEP are isosceles, angle

$$APB = EPD = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ$$

The obtuse angle BPD is therefore

$$360^\circ - (66^\circ + 60^\circ + 66^\circ) = 168^\circ$$