2022-2023 Senior Mathematical Olympiad

Final Round Examination (Grades 9, 10 and 11) - Solutions

1. **Soln**: Let the amount being shared be $\$x$. The three dollar shares are

$$\begin{aligned} \text{Altiman} &: 100+\frac{1}{3}(x-100)=\frac{1}{3}(x+200) \\ \text{Britannie} &: 600+\frac{1}{3}\left(x-\frac{1}{3}(x+200)-600\right)=\frac{2}{9}x+\frac{3400}{9} \\ \text{Cecil} &: 4000 \end{aligned}$$

Since Cecil's amount is the remainder, it represents $x-\left(\frac{1}{3}(x+200)+\frac{2}{9}x+\frac{3400}{9}\right)=\frac{4}{9}x-\frac{4000}{9}$. So

$$\frac{4}{9}x-\frac{4000}{9}=4000$$

$4x-4000=36000$. From this, $4x=40000$ giving $x=\frac{40000}{4}=10\ 000$.

Alternative Solution: After Britannie receives \$600, Cecil's amount represent $\frac{2}{3}$ of what is left which is \$4000. So Britannie's amount is $\$600+\$2000=\$2600$. Solving

$$\frac{2}{9}x+\frac{3400}{9}=2600$$

gives $x=10\,000$.

2. **Soln**: The area of the bigger square is n^2 cm^2 and so the number of tiles available is $T=n^2+92$. After the increase, the area of the bigger square is $(n+2)^2=n^2+4n+4$ and $T=n^2+4n+4-100=n^2+4n-96$. So

$$n^2+92=n^2+4n-96$$

This gives $4n=188$ or $n=\frac{188}{4}=47$. The number of tiles available is $T=47^2+92=2301$.

3. **Soln**: If the number is divisible by 12 it must be divisible by both 3 and 4. It is divisible by 4 if the two digit number $k2$ is divisible by 4. The choices of k are $1,3,5,7,9$. If the number is divisible by 3 the sum of the digits must be divisible by 3. The sum of the digits is $16+k$. The choices of k are $2,5,8$. For the number to be divisible by 12, the only possible value of k is 5 corresponding to 27552.

4. **Soln**:

   (i) If we put $x = 5$ we get $f(5) - f(4) = 20 - 9 = 11$. So

   $$f(4) = f(5) - 11 = 18 - 11 = 7$$

   Similarly, $f(4) - f(3) = 16 - 9 = 7$. So

   $$f(3) = f(4) - 7 = 7 - 7 = 0.$$

   (ii) For $f(x) = 2x^2 + px + q$, $f(3) = 0$ and $f(4) = 7$ implies

   $$\begin{aligned} 18 + 3p + q &= 0 \\ 32 + 4p + q &= 7 \end{aligned}$$

   Subtracting, $14 + p = 7$. This gives $p = -7$ (and $q = 3$).

5. **Soln**: It is easy to show that ΔRAP, ΔPBQ and ΔQCR are all congruent (side-angle-side) and so $PQ = QR = RP$. Therefore ΔPQR is equilateral. Using the cosine rule in ΔRAP, we can find PR:

   $$\begin{aligned} PR^2 &= PA^2 + AR^2 - 2(PA)(AR)\cos(60^\circ) \\ PR^2 &= 1^2 + 3^2 - 2(1)(3)\cos(60^\circ) \\ &= 1^2 + 3^2 - 3 = 7 \end{aligned}$$

   So $PR = \sqrt{7}$ and the area of triangle PQR is

   $$\frac{1}{2}(PQ)(QR)\sin 60^\circ = \frac{1}{2}\sqrt{7} \cdot \sqrt{7} \cdot \frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{4} \text{ cm}^2$$

6. **Soln**: The *layer sum* of $wxyz$ equals 2024 and so

   $$1000w + 100x + 10y + z + 100x + 10y + z + 10y + z + z = 1000w + 200x + 30y + 4z$$

   So $1000w + 200x + 30y + 4z = 2024$

   Now w cannot be 3 or greater and so, $w = 1$ or $w = 2$. If $w = 2$ then $x = y = 0$ and $4z = 24$ giving $z = 6$. So one possible solution is 2006.

   If $w = 1$ then $200x + 30y + 4z = 1024$. Clearly $z = 1$ or $z = 6$. These give

   $$200x + 30y = 1020 \text{ or } 200x + 30y = 1000$$

   That is,

   $$20x + 3y = 102 \text{ or } 20x + 3y = 100$$

Clearly $x = 4$ or $x = 5$. When $x = 4$, no solution exist and when $x = 5$, the second equation gives $y = 0$. The number is therefore 1506. Here

$$1506 + 506 + 06 + 6 = 2024$$

The two solutions are 2006 and 1506.

7. **Soln**: Let the numbers be $m-2, m-1, m.m+1$ and $m+2$

$$(m-2)^2 + (m-1)^2 + m^2 + (m+1)^2 + (m+2)^2 = 1815$$

and so

$$\begin{aligned} m^2 - 4m + 4 + m^2 - 2m + 1 + m^2 + m^2 + 2m + 1 + m^2 + 4m + 4 &= 1815 \\ 5m^2 + 10 &= 1815 \end{aligned}$$

This gives $5m^2 = 1805$ and so $m^2 = 361$. It follows that $m = \sqrt{361} = 19$. The integers are $17, 18, 19, 20, 21$.

8. **Soln**: Expanding,

$$(x^2 + rx + s)(x^2 + tx + u) = x^4 + (t+r)\,x^3 + (u + s + tr)\,x^2 + (ur + st)\,x + su$$

Now

$$x^4 + (t+r)\,x^3 + (u + s + tr)\,x^2 + (ur + st)\,x + su = x^4 + 2023x^2 + N$$

means that

$$\begin{aligned} t + r &= 0 \\ u + s + tr &= 2023 \\ ur + st &= 0 \\ \text{and } su &= N \end{aligned}$$

So $t = -r$ and so

$$\begin{aligned} u + s - r^2 &= 2023 \\ ur - sr &= r\,(u - s) = 0 \\ \text{and } su &= N \end{aligned}$$

Because $r \neq 0$, $u - s = 0$ or $u = s$. Therefore

$$\begin{aligned} r^2 &= 2u - 2023 \\ \text{and } u^2 &= N \end{aligned}$$

N is least when u is least. This occurs when $r^2 = 1$ corresponding with $u = 1012$. The least N is $u^2 = 1012^2 = 1024\,144$.

Note: Here

$$(x^2 + x + 1012)(x^2 - x + 1012) = x^4 + 2023x^2 + 1024\,144$$