2022-2023 Senior Mathematical Olympiad

Round Two Examination (Grades 7 and 8) - Solutions

SECTION A

1. **Soln**: **(C)** We want $\frac{1}{2}\left(1+\frac{1}{100}\right)=\frac{1}{2}+\frac{1}{200}=0.5+0.005=0.505$

2. **Soln**: **(E)** From the information given, 30 are not yellow and $\frac{1}{3}(30)=10$ are red. So 20 are are neither yellow nor red. $\frac{3}{4}$ of this which is 15 are white. The corresponding fraction is $\frac{15}{60}=1/4$.

3. **Soln**: **(B)** Let s, m, ℓ be the number of pages a small, median and large cartridge can print. We have $3s=2m$ and $3m=2\ell$. From this, $9s=6m$ and $6m=4\ell$. Therefore $9s=4\ell$ and $\ell=\frac{9}{4}s=\frac{9}{4}\times 600=1350$.

4. **Soln**: **(E)** Let $\angle PSQ=x^\circ$ so that $\angle PQS=5x^\circ$. Therefore $x^\circ+5x^\circ=6x^\circ=90^\circ$ and $x=15$. Now $\angle QSR=\angle PQS=5x^\circ=5\times 15^\circ=75^\circ$.

5. **Soln**: **(B)** $50=12\times 4+2$ and so 50 months is approximately 4 years and 2 months (or 4 years and 8 weeks) $58=52\times 1+6$ and so 58 weeks is approximately 1 year and 6 weeks (or 1 year and 42 days). So Alfred's present age is approximately $50+4+1=55$ years and $50+42=92$ days. So on his next birthday, he will be 56 years old.

6. **Soln**: **(D)** Presently Sarah wins $\frac{49}{100}\times 500=245$ games. If she plays x more games to get to 50%, then
$$\frac{245+x}{500+x}=\frac{1}{2}$$
This gives $2x+490=500+x$ or $x=10$.

7. **Soln**: **(A)** Let the length of the square be $x+y=1$. The area of each of the rectangle is therefore $\frac{1}{4}$ and one side of the lowest rectangle is therefore $\frac{1}{4}$. The area of the other rectangles are therefore $\frac{3}{4}y$ and $\frac{3}{8}x$. Solving $\frac{3}{8}x=\frac{1}{4}$. This gives $x=2/3$.

8. **Soln**: **(C)** If the sum of two primes is prime then one of the prime must be 2 because odd + odd is even which is not prime. The two prime numbers must take the form $p, p+2$. Similarly, the three primes added must ALL be odd. $3+5+7=15$ is not prime. $3+5+11=19$ and $19=17+2$.

9. **Soln**: **(B)** 16 rows $(1,3,\ldots,31)$ has 16 black squares and 15 black squares in the other 15 rows $(2,4,\ldots,30)$. The total number of black squares is therefore $16^2+15^2=256+225=481$.

10. **Soln**: **(E)** Because PR is a diameter, the angles $\angle PQR$ and $\angle PSR$ are both right angles. By Pythagoras' Theorem

$$PR^2 = PQ^2 + QR^2 \text{ and } PR^2 = PS^2 + RS^2.$$

Hence $PR^2 = 60^2 + 25^2 = 4225$. Also $PS^2 = PR^2 - RS^2 = 4225 - 52^2 = 1521 = 39^2$. So $PS = 39$.

SECTION B

For each question, provide a complete solution by showing all your workings.

1. **Solution**

The units digit of powers of 3 are $3, 9, 7, 1, 3, 9, 7, 1, \cdots$ For example, $3^7 = 2187$. The cycle length is 4. Now $2022 = 505 \times 4 + 2$ and so the 3^{2020} ends in 1 and 3^{2022} ends in 9 and 3^{2023} ends in 7. Therefore the units digit of $3^{2022} + 3^{2023}$ is 6 $(9 + 7 = 16)$.

2. **Solution**

Let the cost of a Pizza, Chilli and a Burger be p, c and b respectively. We have

$$\begin{aligned} 2p + 3c + 4b &= 5300 \\ 5p + 6c + 7b &= 10700 \end{aligned}$$

Multiplying the first equation by 2 and subtracting from the second

$$\begin{aligned} 4p + 6c + 8b &= 10600 \\ 5p + 6c + 7b &= 10700 \end{aligned}$$

gives $p - b = 100$. So $p = 100 + b$. The answer is \$100.

3. **Solution**

From the right most column $A + B = 10$ (because C is the units). 1 is carried and so $1 + A + C = 10$ or $1 + A + C = 20$ (not possible).

So

$$\begin{aligned} A + B &= 10 \\ A + C &= 9 \end{aligned}$$

Finally, the first column says $A = 1$. This results in $B = 9$ and $C = 8$.

The sum is $198 + 981 + 819 = 1998$.

4. **Solution**

Let x represent the length of each side of the octagon, which is also the length of the hypotenuse of each of the right triangles. If ℓ is the length of the legs of the right triangles then $\ell^2 + \ell^2 = x^2$ and so $2\ell^2 = x^2$ and $\ell = \frac{x}{\sqrt{2}}$. The length of the square is

$$2\ell + x = 2\frac{x}{\sqrt{2}} + x = x\left(\sqrt{2} - 1\right) = 10$$

From this,

$$x = \frac{10}{\sqrt{2}+1} = \frac{10}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = 10\left(\sqrt{2}-1\right).$$

5. **Solution**

None of the digits a, b, c, d, e and f may be 0 since this would force two of the others to be equal. Also $d \geq 3$, $e \geq 3$ and $7 \leq f \leq 9$.

Now $f = 7$ means that the d, e pair is $\{3, 4\}$. This is not possible because $4 = 2 + 2$ or $4 = 1 + 3$. So $8 \leq f \leq 9$.

Case $f = 8$. The d, e pairs are$\{3, 5\}$.In this case, the (a, b, c) triplets are $(2, 1, 4)$ and $(4, 1, 2)$.

Case $f = 9$. The d, e pairs are$\{3, 6\}$ and $\{4, 5\}$. For the d, e pair $\{3, 6\}$, the (a, b, c) triplets are $(2, 1, 5)$, $(5, 1, 2)$, $(1, 2, 4)$, $(4, 2, 1)$. For the d, e pair $\{4, 5\}$, the (a, b, c) triplets are $(1, 3, 2)$ and $(2, 3, 1)$.

The solutions are

$$(2,1,4,3,5,8), (4,1,2,5,3,8),\ (2,1,5,3,6,9), (5,1,2,6,3,9),$$
$$(1,2,4,3,6,9), (4,2,1,6,3,9), (1,3,2,4,5,9), (2,3,1,5,4,9).$$

Note that the equations imply $a + 2b + c = f$.