2022-2023 Senior Mathematical Olympiad

Round Two Examination (Grades 9,10 and 11) - Solutions

1. **Soln**: **(E)** To get the smallest possible number we must use the four smallest digits, $0, 1, 2$ and 3. A four digit number cannot begin with a 0. So we must put the next smallest digit, 1, in the thousands place. The hundreds digit must be the smallest remaining digit, 0. Similarly the tens digit must be 2 and the units digit must be 3. The required number is 1023

2. **Soln**: **(B)** If x is the sum of all the positive numbers and y is the sum of all the negative numbers then $x + y = 100$ and

   $x - y = 123 + 45 + 67 + 89 = 324$. Adding these two equations $2x = 424$ and $x = 212$ (which means $y = -112$). A look at the numbers suggest there are 2 negative numbers -45 and -67. So $p = 1$ and $m = 2$. The value of $p - m$ is -1.

3. **Soln**: **(A)** Let n be the two digit number. From the information $n = 3k + 1$ and $n = 4p + 2$ where n and p are positive integers. So $k = \dfrac{4p+1}{3}$ which are integers for $p = 2, 5, 8, 11, 14, 17, 20, 23$ etc. For $p = 2, n = 10$ (the first) and for $p = 23, n = 94$ (the last). The total is 8.

4. **Soln**: **(E)** Using the fact that $\sqrt{a} = a^{1/2}$ and $\left(a^b\right)^c = a^{bc}$, we have

   $$\sqrt{9^{16x^2}} = \left(9^{16x^2}\right)^{1/2} = 9^{\frac{1}{2}\times 16x^2} = 9^{8x^2}$$

   The others are not correct (Put $x = 1$ to illustrate)

5. **Soln**: **(D)** Presently Sarah wins $\frac{49}{100} \times 500 = 245$ games. If she plays x more games to get to 50%, then

   $$\frac{245 + x}{500 + x} = \frac{1}{2}$$

   This gives $2x + 490 = 500 + x$ or $x = 10$.

6. **Soln**: **(E)** Because PR is a diameter, the angles $\angle PQR$ and $\angle PSR$ are both right angles. By Pythagoras' Theorem

   $$PR^2 = PQ^2 + QR^2 \text{ and } PR^2 = PS^2 + RS^2.$$

   Hence $PR^2 = 60^2 + 25^2 = 4225$. Also $PS^2 = PR^2 - RS^2 = 4225 - 52^2 = 1521 = 39^2$. So $PS = 39$.

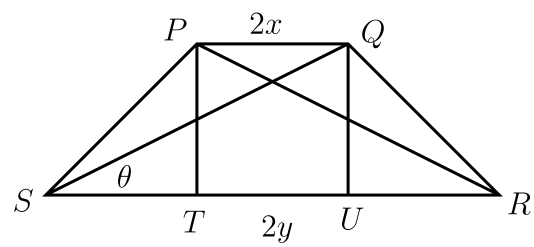
7. **Soln**: **(A)** If the first or second digit is changed then the number ends in 0 and so will be divisible by 10 and hence will not be prime. If the last digit is changed to an even number the resulting number will not be prime because 2 will be a factor. The only choices left are $1, 3, 5, 7, 9$. If the last digit is changed to 1 or 7 the number will be divisible by 3 (because the sum of the digits is divisible by 3) and so will not be prime. If the last digit is 5 the number will be divisible by 5 and will not be prime. Now $203 = 7 \times 29 =$ and $209 = 11 \times 19$ and so they are not prime. So none are prime.

8. **Soln**: **(D)** Let D and E be the point of contacts of the tangent line with the larger and smaller circles respectively. Triangles CBE and CAD are similar. If $BC = x$, then $AC = 5 + 3 + x = 8 + x$. From the similarity of the two triangles, $\dfrac{AD}{AC} = \dfrac{BE}{BC}$ and so

$$\frac{5}{8+x} = \frac{3}{x}.$$

This gives $5x = 24 + 3x$ and so $x = 12$. The length of BC is $x = 12$.

9. **Soln**: **(E)** We label the vertices of the trapezium P, Q, R and S as shown, and we let T, U be the points where the perpendiculars from P, Q, respectively meet the line RS.



Since the diagonals SQ and PR are equal and $PT = QU$. So the right-angled triangles QUS and PTR are congruent. So $SU = TR$ and hence $ST = UR$. Now $ST + UR = SR - TU = SR - PQ = 2y - 2x$. Hence $ST = y - x$ and therefore $SU = ST + TU = ST + PQ = (y-x) + 2x = x + y$. From the right-angled triangle QSU we have $\dfrac{SU}{SQ} = \cos\theta$. Hence $SQ = \dfrac{SU}{\cos\theta} = \dfrac{x+y}{\cos\theta}$.

10. **Soln**: **(A)** The sum of the interior angles of a triangle is $180°$ and so

$$\begin{aligned} 5x + 3y + 3x + 20 + 10y + 30 &= 180 \\ 8x + 13y &= 130 \end{aligned}$$

This gives $8x = 13\,(10 - y)$. This suggests that $x = 13$ and $10 - y = 8$. So $x = 13$ and $y = 2$ resulting in $x + y = 15$.

The three angles are $71°, 59°$ and $50°$.

SECTION B

1. **Solution**

Filling in angles, $\angle BCE = \angle ECA = \frac{180-(72+84)}{2} = 12°$. Also, $\angle ABD = 180 - 72 = 108°$ and $\angle BED = \frac{180-84}{2} = 48°$ with $\angle BDA = 180 - (48+108) = 24°$. It is now evident that ΔACD is isosceles with $AD = AC$. $\angle CEA = 180 - 96 = 84°$ and so ΔACE is also isosceles with $AC = CE$. So $AD = CE$.

2. **Solution**

Let $N = uv$ be the two digit number. $P(N) + S(N) = u + v + u \cdot v$.

$N = 10u + v$ and so v is the units digit of N

Since $N = P(N) + S(N)$,

$$10u + v = u + v + u \cdot v$$

and so $10u = u + u \cdot v$ and $9u = uv$. This gives $v = 9$ (since $u \neq 0$) which is the units digit of N.

3. **Solution**

None of the digits a, b, c, d, e and f may be 0 since this would force two of the others to be equal. Also $d \geq 3$, $e \geq 3$ and $7 \leq f \leq 9$.

Now $f = 7$ means that the d, e pair is $\{3, 4\}$. This is not possible because $4 = 2 + 2$ or $4 = 1 + 3$. So $8 \leq f \leq 9$.

Case $f = 8$. The d, e pairs are$\{3, 5\}$.In this case, the (a, b, c) triplets are $(2, 1, 4)$ and $(4, 1, 2)$.

Case $f = 9$. The d, e pairs are$\{3, 6\}$ and $\{4, 5\}$. For the d, e pair $\{3, 6\}$, the (a, b, c) triplets are $(2, 1, 5)$, $(5, 1, 2)$, $(1, 2, 4)$, $(4, 2, 1)$. For the d, e pair $\{4, 5\}$, the (a, b, c) triplets are $(1, 3, 2)$ and $(2, 3, 1)$.

The solutions are

$$(2, 1, 4, 3, 5, 8), (4, 1, 2, 5, 3, 8),\ (2, 1, 5, 3, 6, 9), (5, 1, 2, 6, 3, 9),$$
$$(1, 2, 4, 3, 6, 9), (4, 2, 1, 6, 3, 9), (1, 3, 2, 4, 5, 9), (2, 3, 1, 5, 4, 9).$$

Note that the equations imply $a + 2b + c = f$.

4. **Solution**

Since $OP = OR$ are radii, triangle OPR is isosceles with $\angle RPO = \angle PRO$. From the exterior angle theorem, they add to $5x°$ and so $\angle RPO = \angle PRO = \frac{5}{2}x°$. By a similar argument, $\angle TQO = \frac{3}{2}x°$. Finally, by the exterior angle theorem, $\angle RSQ = \frac{5}{2}x° + \frac{3}{2}x° = 4x°$.

5. **Solution**

Let the first two terms in the sequence be a, b. The third to eighth terms are

$$a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b$$

and $8a+13b=400$. From this $13b=400-8a=8\left(50-a\right)$.

This suggests that $50-a$ is a multiple of 13. This gives $a=50-13=37$ or $50-26=24$ or $50-39=11$. Using $b=\dfrac{8\left(50-a\right)}{13}$, we compute the following table:

a	b	$a+b$
37	8	45
24	16	40
11	24	35

The third term is $a+b$ which has maximum value of 45.