2024-2025 Senior Mathematical Olympiad

Final Round Examination (Grades 7 and 8)
SOLUTIONS

1. **Solution:** Sue pays a minimum amount when she buys them in batches of 7. Since $2025 = 7 \times 289 + 2$, she can buy 289 batches of seven beads and then buy 2 single beads. The total spent is, in dollars, $289 \times 60 + 2 \times 10 = 173\,40 + 20$. That is, $\$17,360$.

2. **Solution:** After the first cycle of cuts, he has $8 = 1 + 7$ pieces. After the second, he has $7 + 8 = 2 \times 7 + 1$, after the 3rd, he has $7 + 7 + 8 = 3 \times 7 + 1$. After each cycle of cuts the total is increased by 7 and after the nth cycle, the number of pieces is $7n + 1$.

3. **Solution:** Let the radii be r and $3r$ the area of the shaded region is $\pi\left(3r\right)^2 - \pi r^2 = 8\pi r^2$. The ratio is $8\pi r^2 : \pi r^2$ or $8 : 1$.

4. **Solution:** The number of green marbles to start is $2\% \times 2000 = \frac{2}{100} \times 2000 = 40$ and the number of red marbles is 1960. Let n be the number of red marbles removed. We have, $\dfrac{40}{2000 - n} = \dfrac{20}{100} = \dfrac{1}{5}$. This gives $200 = 2000 - n$. So $n = 1800$.

5. **Solution:** Let the dimensions of L be ℓ by $4w$ so that its perimeter is $2\ell + 8w$ and its area is $4\ell w$. The perimeter and area of one of the smaller rectangle are $2\ell + 2w$ and ℓw. We therefore have $2\ell + 8w = 2\ell + 2w + 18$ and $4\ell w = \ell w + 18$. So $6w = 18$ and $3\ell w = 18$. These give $w = 3$ and $\ell = 2$. The perimeter of L is $2\ell + 8w = 2\left(2\right) + 8\left(3\right) = 28$.

6. **Solution:** Since $9^2 = 81 > 65$, and 65 is not a perfect square, we know that $m, n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Since $5^2 = 25$ and $6^2 = 36$, the larger of the two digits must be at least 6. We only need to check with digits $6, 7, 8$. **Case** $m = 6 : 65 - 6^2 = 29$ and this is not a perfect square. **Case** $m = 7 : 65 - 7^2 = 16 = 4^2$. Therefore, 74 and 47 are easy numbers. **Case** $m = 8 : 65 - 8^2 = 1 = 1^2$. Therefore, 81 and 18 are easy numbers. The set of all easy numbers are $47, 74, 81$ and 18.

7. **Solution:** Let x km be the distance from Anya's home where they meet. The distance travelled by Britany is then $(25 - x)$ km. The time of travel in hours for both girls is t where $x = 8t$ and $(25 - x) = 7t$. Combining these, $(25 - 8t) = 7t$. If follows that $25 = 15t$ and $t = \frac{5}{3}$ hour $= 1\frac{2}{3}$ hour $=$ 1 hour 40 minutes. Since the time of travel is 1 hour 40 minutes, they should both leave their home at 11:20 am.

8. **Solution:** As a product of prime numbers, $7200 = 72 \times 100 = (2 \times 2 \times 2 \times 3 \times 3) \times (2 \times 5 \times 2 \times 5)$. So $7200 = 2^5 \times 3^2 \times 5^2$. This can be written as $2^{2c} \times 2^{5-2c} \times 3^2 \times 5^2$. That is, $2^{5-2c} \times 3^2 \times 4^c \times 5^2$. Now, $5 - 2c + 2 + c + 2 = 7$. This gives $c = 2$.

9. **Solution:** In a perfect square, every prime factor occurs an even number of times. Now, $120 = 2^3 \cdot 3 \cdot 5$. Thus, if the product $120m$ is to be a perfect square, then the smallest possible value of m is $m = 2 \cdot 3 \cdot 5 = 30$. In this case, $120m = 120 \times 30 = 3600 = \left(2^2 \cdot 3 \cdot 5\right)^2 = 60^2$.

10. **Solution:** The number of painted faces of the cubes is either $1, 2$ or 3.

    Case 3 faces painted: These are at the 8 vertices and the number of cubes is 8.

    Case 2 faces painted: These are at the 12 edges that are not vertices. The number is $4(5-2) + 4(8-2) + 4(13-2) = 80$.

    Case 2 face painted: These are at the 6 faces that are not edges or vertices. The number is $2(5-2)(8-2) + 2(5-2)(13-2) + 2(8-2)(13-2) = 234$.

    The total is $8 + 80 + 234 = 322$.

    Alternatively: The number of cubes in the prism is $5 \times 8 \times 13 = 520$. The number with no paint are the unexposed cubes. The number is $(5-2)(8-2)(13-2) = 198$. Therefore the number of painted cubes is $520 - 198 = 322$.