2024-2025 Senior Mathematical Olympiad

Final Round Examination (Grades 9, 10 and 11)

SOLUTIONS

1. **Solution:** As a product of prime numbers, $7200 = 72 \times 100 = (2 \times 2 \times 2 \times 3 \times 3) \times (2 \times 5 \times 2 \times 5)$. So $7200 = 2^5 \times 3^2 \times 5^2$. This can be written as $2^{2c} \times 2^{5-2c} \times 3^2 \times 5^2$. That is, $2^{5-2c} \times 3^2 \times 4^c \times 5^2$. Now, $5 - 2c + 2 + c + 2 = 7$. This gives $c = 2$.

2. **Solution:** Let the dimensions of L be ℓ by $4w$ so that its perimeter is $2\ell + 8w$ and its area is $4\ell w$. The perimeter and area of one of the smaller rectangle are $2\ell + 2w$ and ℓw. We therefore have $2\ell + 8w = 2\ell + 2w + 18$ and $4\ell w = \ell w + 18$. So $6w = 18$ and $3\ell w = 18$. These give $w = 3$ and $\ell = 2$. The perimeter of L is $2\ell + 8w = 2(2) + 8(3) = 28$.

3. **Solution:** $101! = 101 \times 100 \times 99!$ and so

$$\frac{99!}{101! - 99!} = \frac{99!}{101 \times 100 \times 99! - 99!} = \frac{99!}{99!\,(101 \times 100 - 1)}.$$

This is equal to $(101 \times 100 - 1) = 10\,099$

$$\frac{1}{(101 \times 100 - 1)} = \frac{1}{10100 - 1} = \frac{1}{10\,099}$$

So $n = 10\,099$.

4. **Solution:** Let x kmh^{-1} be his average speed for the journey. The planned average speed is $(x - 5)$ kmh^{-1} and equating times,

$$\frac{210}{x-5} = \frac{210}{x} + 1.$$

Multiplying by $x(x-5)$, we have

$$210x - 210(x-5) - x(x-5) = 0$$

This gives $x(x-5) - 1050 = 0$. That is, $x^2 - 5x - 1050 = 0$. That is, $(x - 35)(x + 30) = 0$. So $x = 35$ and the speed is 35 kmh^{-1}.

5. **Solution:** First $165 = 5 \times 33 = 3 \times 5 \times 11$ and so $aa4ab$ is divisible by $3, 5$ and 11. Since 5 divides the number, $b = 0$ or $b = 5$. Since 3 divides $aa4ab$, $3a + 4 + b$ is divisible by 3. This means that $4 + b$ is divisible by 3 and so $b \neq 0$ and $b = 5$. Finally, since 11 divides $aa4a5$, 11 divides $a - a + 4 - a + 5 = 9 - a$. This means that $a = 9$ and the number is 99495.

6. **Solution:** Let the number of males enrolled and females enrolled last year be m and f respectively so that the total is $m+f$. This year, the number of females is $f^* = 1.1f$. The current number of males is m^* where $m^* + f^* = m + f$. That is, $m^* + \frac{11}{10}f = m + f$ with $m^* = m - \frac{1}{10}f$. Also $\dfrac{m}{f} = \dfrac{19}{25}$ or $f = \dfrac{25}{19}m$. So $f^* = \dfrac{11}{10} \times \dfrac{25}{19}m$ and $m^* = m - \dfrac{1}{10} \times \dfrac{25}{19}m$. The required ratio is therefore $\dfrac{m^*}{f^*} = \dfrac{m - \dfrac{1}{10} \times \dfrac{25}{19}m}{\dfrac{11}{10} \times \dfrac{25}{19}m} = \dfrac{1 - \dfrac{25}{190}}{\dfrac{275}{190}}$. This is $\dfrac{190-25}{275} = \dfrac{165}{275} = \dfrac{3}{5}$. The ratio is $3 : 5$.

7. **Solution:** Suppose $75 = a + a + 1$, then $2a = 74$ and $a = 37$. So $37 + 38 = 75$. Suppose $75 = b - 1 + b + b + 1$, then $75 = 3b$ and $b = 25$. So $24 + 25 + 26 = 75$. Suppose $75 = c + c + 1 + c + 2 + c + 3$, then $75 = 4c + 6$. No integer solution exists here. Suppose $75 = d - 2 + d - 1 + d + d + 1 + d + 2$, then $75 = 5d$ and $d = 15$. So $13 + 14 + 15 + 16 + 17 = 75$. Suppose $75 = e + e + 1 + e + 2 + e + 3 + e + 4 + e + 5$, then $75 = 6e + 15$. That is , $6e = 60$ and $e = 10$. Therefore $10 + 11 + 12 + 13 + 14 + 15 = 75$. Since $7f = 75$ has no integer solution, 75 cannot be written as the sum of 7 consecutive positive integers. Suppose $75 = g + g + 1 + g + 2 + g + 3 + g + 4 + g + 5 + g + 6 + g + 7$, then $75 = 8g + 28$. No integer solution exists here. $9h = 75$ has no integer solution, 75 cannot be written as the sum of 9 consecutive positive integers. $75 = 10g + 45$ implies $g = 3$ and so $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 75$. Since $11k = 75$ has no integer solution, 75 cannot be written as the sum of 11 consecutive positive integers. Since the sum of the first 12 positive integers is 78, the sum of any 12 or more consecutive positive integers is at least 78. So the possible values of m are $2, 3, 5, 6$ and 10.

8. **Solution:** Let n be the number of men. We require

$$\underbrace{1 + 1 + 1 + \cdots 1}_{n} + 1 + \underbrace{2 + 2 + 2 + \cdots + 2}_{n-1} \geq 2025$$

That is, $n + 1 + 2(n-1) \geq 2025$. That is, $3n - 1 \geq 2025$. This gives $n \geq 675\frac{1}{3}$. Take $n = 676$.

9. **Solution:** The amount paid in error was $\$abX00$ where X is a digit. We therefore have $\$abX00 - \$ab00 = \$64700$ or $abX = ab + 647$. This means that

$$a \times 100 + b \times 10 + X = a \times 10 + b + 6 \times 100 + 4 \times 10 + 7$$

That is,

$$a \times 100 + (b - a) \times 10 + X - b = 647 \text{ or } 90a + 9b + X = 647$$

That is, $9(10a + b) + X = 647$. So X is the remainder after dividing by 9. Since $647 = 9 \times 71 + 8$, $X = 8$ and $10a + b = 71$. So $a = 7$ and $b = 1$. Mrs Banks should have paid $\$ab00$ or \$7100 (but instead paid \$71800).

10. **Solution:** Let x m be the distance traveled due east, and y m, the distance travel due south.

$$\begin{aligned} x &= 1-3+5-7+9-11+\cdots+37-39+41 \\ &= (1-3)+(5-7)+(9-11)+\cdots+(37-39)+41 = 10\,(-2)+41 = 21 \\ \text{and } y &= (-2+4)+(-6+8)+(-10+12)+\cdots+(-38+40) = 10\,(2) = 20 \end{aligned}$$

Using Pythagoras, theorem,

$$OP = \sqrt{x^2+y^2} = \sqrt{21^2+20^2} = \sqrt{841} = 29 \text{ metres.}$$