The University of the West Indies, Mona

presents

2025-2026 Senior Mathematical Olympiad

Final Round Examination (Grades 9, 10 and 11) - SOLUTIONS

1. **Soln**: The expression is of the form $\frac{(x+y)^2}{x^2-y^2} = \frac{(x+y)^2}{(x+y)(x-y)} = \frac{x+y}{x-y}$. Putting $x = 2026$ and $y = 2024$, the result is $\frac{2026+2024}{2026-2024} = 2025$.

2. **Soln**: Factorizing the numerator and cancelling,

$$\frac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2026^2}\right)}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2026}\right)} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{2026}\right)$$

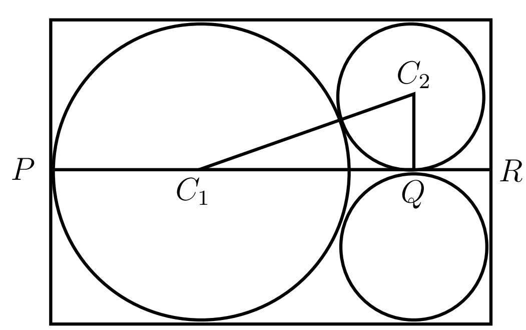
This is

$$\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\cdots\cdot\frac{2026}{2025}\cdot\frac{2027}{2026}$$

After cancelling the result is $\frac{2027}{2} = 1013.5$

3. **Soln**: From 1 to 9, the number of digits is 9. From 10 to 99, the number of digits is $2 \times 90 = 180$. From 100 to 999, the number of digits is $3 \times 900 = 2700$. This is bigger than 2025 and so n is a three digit number. Since $2025 - 189 = 1836$, the number of three digit numbers is $\frac{1836}{3} = 612$. That is 100 to 711 and so $n = 711$.

4. **Soln**: Let C_1 denote the centre of the larger circle and C_2 the centre of the upper smaller circle. Let P, Q and R are as shown on the diagram below:



The length ℓ is $\ell = PC_1 + C_1Q + QR$, where $PC_1 = 2, QR = QC_2 = 1$ and so $\ell = 3 + C_1Q$. Of course $C_1C_2 = 2 + 1 = 3$ and using Pythagoras' Theorem, $C_1Q^2 = 3^2 - 1^2 = 8$ and so $C_1Q = \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$. It now follows that $\ell = 3 + 2\sqrt{2}$.

5. **Soln**: The number of such four-digit numbers is $5 \times 4 \times 3 \times 2 = 120$. Of these, the number that includes the digit 1 is $4\,(1 \times 4 \times 3 \times 2) = 96$ and the number that does not contain 1 is $120 - 96 = 24$. The sum of these 24 numbers is

$$6\,(2+3+4+5)\,(1000+100+10+1) = 93\,324.$$

Similarly the sum of the numbers that does not contain 2 is

$$6\,(1+3+4+5)\,(1000+100+10+1) = 86\,658.$$

The sum of the numbers that does not contain 3, 4, 5 are respectively

$$\begin{aligned} 6\,(1+2+4+5)\,(1000+100+10+1) &= 79\,992, \\ 6\,(1+2+3+5)\,(1000+100+10+1) &= 73\,326 \\ \text{and } 6\,(1+2+3+4)\,(1000+100+10+1) &= 66\,660. \end{aligned}$$

The grand total is

$$6\,(1000+100+10+1) \times 4 \times (1+2+3+4+5) = 399\,960.$$

6. **Soln**: Since 23 is a factor of $3a + 2b$, $23k = 3a + 2b$ where k is an integer. Now,

$$\begin{aligned} 17a + 19b &= 23a - 6a + 23b - 4b = 23\,(a+b) - 2\,(3a+2b) \\ &= 23\,(a+b) - 2 \times 23k = 23\,(a+b-2k)\,. \end{aligned}$$

Therefore 23 is a factor of $17a + 19b$.

7. **Soln**:

   (a) $3^{10} = 3^6 \times 3^4 = (\cdots 29)\,81\times = \cdots 49$. The last two digits is 49

   (b) $3^{20} = (3^{10})^2 = (\cdots 49)^2 = \cdots 01$. The last two digits is 01

   (b) Now $2026 = 101 \times 20 + 6 = 2026$ and so

   $$3^{2026} = \left(3^{20}\right)^{101} \times 3^6 = (\cdots 01)^{101}\,(\cdots 29) = (\cdots 01)\,(\cdots 29) = 29.$$

   The last two digits of 3^{2026} is 29.

8. **Soln**: Let the dimension of the tiles be $\ell \times \ell$ cm^2. ℓ has to be a factor of 135 and 210. In fact, ℓ should be the highest common factor of 135 and 210 which is 15. Since $\frac{210}{15} = 14$ and $\frac{135}{15} = 9$. The number of tiles to be used is $14 \times 9 = 126$.

9. **Soln**: The palindromes take one of the following forms $abcba, aabaa, abbba, ababa$ and $aaaaa$.

   Case $abcba$: Conditionally, there are 9 choices for a, 9 choices for b and 8 choices for c. The total is $9 \times 9 \times 8 = 648$

   Case $aabaa$: Conditionally, there are 9 choices for the a, 9 choices for the b. The total is $9 \times 9 = 81$

   Case $abbba$: Conditionally, there are 9 choices for the a, 9 choices for the b. The total is $9 \times 9 = 81$

   Case $ababa$: Conditionally, there are 9 choices for the a, 9 choices for the b. The total is $9 \times 9 = 81$

   Case $aaaaa$: Conditionally, there are 9 choices for the a. The total is 9

   The total is $648 + 81 + 81 + 81 + 9 = 900$.

   Alternatively, The palindrome is determined by the first three digits: 9 choices for the first digit, and 10 choices for the second and third digit. The total is $9 \times 10 \times 10 = 900$.

10. **Soln**: We have $\frac{1}{a_{n+1}} = \frac{1 + na_n}{a_n} = n + \frac{1}{a_n}$. Put $n = 1, 2, 3, 4$ to get

$$\begin{aligned}
\frac{1}{a_2} &= 1 + \frac{1}{a_1} = 1 + 1 \\
\frac{1}{a_3} &= 2 + \frac{1}{a_2} = 2 + 1 + 1, \\
\frac{1}{a_4} &= 3 + \frac{1}{a_3} = 3 + 2 + 1 + 1, \\
\frac{1}{a_5} &= 4 + \frac{1}{a_4} = 4 + 3 + 2 + 1 + 1
\end{aligned}$$

   By induction, we can prove that $\frac{1}{a_{n+1}} = 1 + (1 + 2 + 3 + \cdots + n)$.

   It follows that $\frac{1}{a_{n+1}} = 1 + \frac{1}{2}n(n+1)$ and so

$$\frac{1}{a_{2026}} = 1 + \frac{1}{2}(2025)(2026) = 2051\,326.$$