Problem 1 (Equations with the dependent variable missing)

By means of the substitutions
\[ v = \frac{dy}{dt}, \quad \frac{dv}{dt} = \frac{d^2y}{dt^2} \]
solve the following second-order differential equations

1. \( t^2 \frac{d^2y}{dt^2} + 2t \frac{dy}{dt} - 1 = 0, \quad t > 0; \)
2. \( \frac{d^2y}{dt^2} + \frac{dy}{dt} = e^{-t}; \)
3. \( 2t^2 \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^3 = 2t \frac{dy}{dt}, \quad t > 0. \)

Problem 2 (Equations with the independent variable missing)

By means of the substitutions
\[ v = \frac{dy}{dt}, \quad \frac{dv}{dt} = \frac{d^2y}{dt^2}, \quad \frac{dv}{dy} = \frac{d^2y}{dt^2}, \quad \frac{d^2y}{dy} = v \frac{dv}{dy} \]
solve the following second-order differential equations

1. \( y \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^2 = 0; \)
2. \( 2y^2 \frac{d^2y}{dt^2} + 2y \left( \frac{dy}{dt} \right)^2 = 1; \)
3. \( y \frac{d^2y}{dt^2} - \left( \frac{dy}{dt} \right)^3 = 0. \)

Problem 3 (Wronskians)

Compute the Wronskian of the given pair of functions

1. \( y_1(t) = \cos t, \quad y_2(t) = \sin t; \)
2. \( y_1(t) = e^t \sin t, \quad y_2(t) = e^t \cos t; \)
3. \( y_1(\vartheta) = \cos^2 \vartheta, \quad y_2(\vartheta) = 1 + \cos (2\vartheta). \)
Problem 4

In each of the following problems determine the longest interval in which the given initial value problem is certain to have a unique solution. Do not attempt to find the solution.

1. \((t - 1) \frac{d^2y}{dt^2} - 3t \frac{dy}{dt} + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1;\)

2. \(\frac{d^2y}{dt^2} + (\cos t) \frac{dy}{dt} + 3(\ln |t|)y = 0, \quad y(2) = 3, \quad y'(2) = 1;\)

3. \((x - 2) \frac{d^2y}{dx^2} + \frac{dy}{dx} + (x - 2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2.\)

Problem 5

Verify that \(y_1(t) = t^2\) and \(y_2(t) = 1/t\) are two solutions of the differential equation

\[ t^2 \frac{d^2y}{dt^2} - 2y = 0 \quad \text{for} \quad t > 0. \]

Then show that \(c_1t^2 + c_2t^{-1}\) is also a solution of this equation for any \(c_1\) and \(c_2\).

Problem 6

Verify that \(y_1(t) = 1\) and \(y_2(t) = \sqrt{t}\) are solutions of the differential equation

\[ y \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^2 = 0 \quad \text{for} \quad t > 0. \]

Then show that \(c_1t^2 + c_2\sqrt{t}\) is not in general a solution of this equation. Explain why this result does not contradict the existence and uniqueness theorem for linear second order differential equations.

Problem 7

If the Wronskian \(W\) of the functions \(f\) and \(g\) is \(3e^{4t}\), and if \(f(t) = e^{2t}\), find \(g(t)\). This problem should make you realize that it is enough to find a special solution of a second order linear differential equation since a second linearly independent solution can be found by setting up a first order equation for it with the help of the Wronskian and Abel’s formula.
Problem 8

If $W(f, g)$ is the Wronskian of $f$ and $g$, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ in terms of $W(f, g)$.

Problem 9

In each of the following problems verify that the functions $y_1$ and $y_2$ are solutions of the given differential equation. Do they are linearly independent?

1. $x^2 \frac{d^2 y}{dx^2} - x(x + 2) \frac{dy}{dx} + (x + 2)y = 0, x > 0, y_1(x) = x, y_2(x) = xe^x$;

2. $(1 - x \cot x) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0, x \in (0, \pi), y_1(x) = x, y_2(x) = \sin x$.

Problem 10 (Exact equations)

The equation

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

is said to be exact if it can be rewritten in the form

$$\frac{d}{dx} \left( P(x) \frac{dy}{dx} \right) + \frac{d}{dx} (f(x)y) = 0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated immediately and gives rise to a first order linear equation for $y$.

1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$\frac{d^2 P}{dx^2} - \frac{dQ}{dx} + R(x) = 0.$$ 

2. In each of the following problems determine whether the given equation is exact. If so, solve the equation.

(a) $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$;

(b) $x \frac{d^2 y}{dx^2} - (\cos x) \frac{dy}{dx} + (\sin x)y = 0, \quad x > 0.$
Problem 11 (The adjoint equation)

If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)\frac{d^2y}{dx^2} + \mu(x)Q(x)\frac{dy}{dx} + \mu(x)R(x)y = 0$$

can be written in the form

$$\frac{d}{dx}\left(\mu(x)P(x)\frac{dy}{dx}\right) + \frac{d}{dx}(f(x)y) = 0.$$

1. By equating coefficients in these two equations and eliminating $f(x)$, show that the function $\mu(x)$ must satisfy

$$P\frac{d^2\mu}{dx^2} + \left(\frac{dP}{dx} - Q\right)\frac{d\mu}{dx} + \left(\frac{d^2P}{dx^2} - \frac{dQ}{dx} + R\right)\mu = 0.$$

This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations.

2. In each of the following problems find the adjoint of the given differential equations.

(a) Bessel equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \quad \nu \in \mathbb{R}$$

(b) Legendre equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha + 1)y = 0, \quad \alpha \in \mathbb{R}$$

(c) Airy equation

$$\frac{d^2y}{dx^2} - xy = 0.$$

Problem 12

In each of the following problems determine whether the given pair of functions is linearly independent or linearly dependent

1. $f(t) = t^2 + 5t, \quad g(t) = t^2 - 5t$;

2. $f(t) = e^{\lambda t}\cos(\mu t), \quad g(t) = e^{\lambda t}\sin(\mu t), \quad \mu \neq 0, \quad \mu, \lambda \in \mathbb{R}$;

3. $f(t) = 3t, \quad g(t) = |t|$.
Problem 13

The Wronskian of two functions is $W(t) = t \sin^2 t$. Are the functions linearly independent or linearly dependent? Why?

Problem 14

In each of the following problems find the Wronskian of two solutions of the given differential equation without solving the equation

1. Bessel equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0, \quad \nu \in \mathbb{R}$$

2. Legendre equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \alpha(\alpha + 1)y = 0, \quad \alpha \in \mathbb{R}.$$  

Problem 15

If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients $p$ and $q$?

Problem 16

If $f$, $g$, and $h$ are differentiable functions, show that

$$W(fg, fh) = f^2W(g, h).$$

Problem 17

Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$ but are linearly independent on $-1 < t < 1$. Although $f$ and $g$ are linearly independent there, show that $W(f, g)$ is zero for all $t$ in $-1 < t < 1$. Hence, $f$ and $g$ cannot be solutions of an equation $y'' + p(t)y' + q(t)y = 0$ with $p$ and $q$ continuous on $-1 < t < 1$. 