Lecture VIII

Abstract

After a concise review of partial derivatives we give a classification of second order linear partial differential equations (PDEs) with constant coefficients into three different types: elliptic, parabolic and hyperbolic equations. In particular, we introduce the heat equation as an example of a parabolic equation which will be studied in some detail in the next lectures.

Review of partial derivatives

You already know that if \( y(x) \) is a function of one variable \( x \), then the derivative of \( y(x) \) with respect to \( x \) can be thought of as the instantaneous rate of change of \( y \) at \( x \). The derivative, when it exists, can be calculated as follows

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}.
\]

If a function \( u \) is a function of more than one variable, for instance \( u = u(x, y) \), then we need to use partial derivatives to measure the rate of change of \( u \) with respect to each of the variables. For example, the instantaneous rate of change of \( u \) with respect to \( x \) is called the partial derivative of \( u \) with respect to \( x \) and is defined by

\[
\partial_x u = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}.
\]

This is just the derivative of \( u \) with respect to \( x \), assuming that \( y \) is held fixed and in general \( \partial_x u \) is also a function of both \( x \) and \( y \). Similarly, the partial derivative of \( u \) with respect to \( y \) is given by

\[
\partial_y u = \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}.
\]

We will also need second-order partial derivatives and these are defined pretty much the same. For example, \( \partial_{xx} u \) is the instantaneous rate of change of the function \( \partial_x u \) with respect to the variable \( x \), that is

\[
\partial_{xx} u = \frac{\partial^2 u}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\partial_x u(x + \Delta x, y) - \partial_x u(x, y)}{\Delta x}.
\]

The other second order partial derivatives we will need are

\[
\partial_{yy} u = \frac{\partial^2 u}{\partial y^2}.
\]
and the two mixed partial derivatives

$$\partial_{xy} u = \frac{\partial^2 u}{\partial x \partial y}, \quad \partial_{yx} u = \frac{\partial^2 u}{\partial y \partial x}. $$

The following theorem gives a very useful condition under which the second order mixed partial derivatives $\partial_{xy} u$ and $\partial_{yx} u$ coincide.

**Theorem 1 (Clairaut’s theorem)**

Suppose that $u = u(x, y)$ is defined on an open subset $D$ of $\mathbb{R}^2$ without holes such that $\partial_{xy} u$ and $\partial_{yx} u$ are both continuous on $D$. Then,

$$\partial_{xy} u = \partial_{yx} u \quad \forall (x, y) \in D.$$

**Example 1** For the function

$$u(x, y) = \sin (4x) \cos (3y)$$

find all first and second derivatives and show that $u(x, y)$ satisfies the partial differential equation

$$\partial_{xx} u - \frac{16}{9} \partial_{yy} u = 0.$$

The six derivatives are

$$\partial_x u = 4 \cos (4x) \cos (3y), \quad \partial_{xx} u = -16 \sin (4x) \cos (3y),$$

$$\partial_y u = -3 \sin (4x) \sin (3y), \quad \partial_{yy} u = -9 \sin (4x) \cos (3y),$$

$$\partial_{xy} u = -12 \cos (4x) \sin (3y) = \partial_{yx} u.$$

To show that $u$ satisfies the given PDE we substitute $\partial_{xx} u$ and $\partial_{yy} u$ into the PDE and find that

$$\partial_{xx} u - \frac{16}{9} \partial_{yy} u = -16 \sin (4x) \cos (3y) - \frac{16}{9} (-9 \sin (4x) \cos (3y)),$$

$$= -16 \sin (4x) \cos (3y) + 16 \sin (4x) \cos (3y) = 0.$$

Thus, $u$ satisfies the original PDE.

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1Alexis Claude Clairaut (1713-1765) was a French mathematician, astronomer, geophysicist and intellectual.
Classification of linear second order PDEs

All PDEs we will study are linear equations of second order which are a particular case of the more general second order linear equation

\[ A \partial_{x_1 x_1} u + B \partial_{x_1 x_2} u + C \partial_{x_2 x_2} u + D \partial_{x_1} u + E \partial_{x_2} u + F u = G, \tag{1} \]

where the unknown function \( u \) depends only on two independent variables \( x_1 \) and \( x_2 \). Depending on the particular problem modelled by (1) we may have \( x_1 = t \) and \( x_2 = x \) or \( x_1 = x \) and \( x_2 = y \). In general, the coefficients \( A, B, \cdots, G \) can be arbitrary functions of independent variables \( x_1 \) and \( x_2 \). Any PDE that can be put in the form (1) is called linear and if \( G = 0 \) equation (1) represents a homogeneous linear PDE. We will mainly be concerned with homogeneous equations where the coefficients \( A, B, \cdots, F \) are constants. However, you have to know that the classification given here below is valid also in the case that \( A, B, \cdots, F \) do depend on \( x_1 \) and \( x_2 \). Depending on the coefficients \( A, B, \cdots, F \) equation (1) with \( G = 0 \) can be classified in the following three different types

1. (1) is called elliptic if \( B^2 - 4AC < 0 \);
2. (1) is called parabolic if \( B^2 - 4AC = 0 \);
3. (1) is called hyperbolic if \( B^2 - 4AC > 0 \).

Notice that since in general the coefficients \( A, B \) and \( C \) may depend on the variables \( x_1 \) and \( x_2 \) we can have for example the situation where a given PDE is elliptic on a certain region in \( \mathbb{R}^2 \) and parabolic in another portion of the plane. We will see that the methods used to solve a given PDE depend on which type it is. As an example of an elliptic equation we will solve the Laplace equation

\[ \partial_{xx} u + \partial_{yy} u + \partial_{zz} u = 0 \]

in spherical and cylindrical coordinates. The one-dimensional heat equation

\[ \partial_t u = \alpha^2 \partial_{xx} u \]

will be our example of a parabolic equation. Finally, the one-dimensional wave equation

\[ \partial_{tt} u = b^2 \partial_{xx} u \]

will be the hyperbolic example. All three of these PDEs, along with several variations of each, appear in many different fields of Engineering and Physics.
Parabolic partial differential equations

Parabolic partial differential equations belong to a type of second order PDEs describing a wide family of problems in science including heat diffusion, transport processes, ocean acoustic propagation, evolution of quantum systems like the Schrödinger equation\(^2\) and stock option pricing (Black\(^3\)-Scholes\(^4\) equation in Finance). These problems also known as evolution problems, describe a physical or mathematical system with a time variable and which behaves essentially like heat diffusion through a solid. Here below, I prepared a short list of the most famous parabolic equations

1. The one-dimensional heat equation

\[ \partial_t T = \alpha^2 \partial_{xx} T, \]

where \( \alpha \) is a real constant depending on certain physical parameters of the problem at hand. For instance, it describes the temperature \( T(t, x) \) at time \( t \) and position \( x \) along the length of a thin rod with uniform cross sectional area \( A \), density \( \rho \) and length \( L \). The sides of the rod may or may not be insulated. For this particular example, we have

\[ \alpha^2 = \frac{\kappa}{\rho s}, \]

where \( \kappa \) is the thermal conductivity of the rod material and \( s \) is the specific heat of the material in the rod.

2. The one-dimensional Schrödinger equation

\[ i \partial_t \Psi = -\hbar \partial_{xx} \Psi + V(x) \Psi, \quad \hbar = \frac{h}{2m} \]

describes the time evolution of a quantum physical system such as a particle of mass \( m \) immersed in a potential \( V \). Once one solves the above equation for \( \Psi \), the combination

\[ \Psi(t, x)\overline{\Psi(t, x)} = |\Psi(t, x)|^2 \]

gives the probability of finding the particle at the time \( t \) at the position \( x \). Finally, the Planck\(^5\) constant \( \hbar \) is a physical constant reflecting the sizes of quanta in quantum mechanics.

\(^2\)Erwin Rudolph Joseph Alexander Schrödinger (1887-1961) was an Austrian theoretical physicist who received the Nobel Prize in Physics for his fundamental contributions in Quantum Mechanics.

\(^3\)Fischer Sheffey Black (1938-1995) was an American economist

\(^4\)Myron Samuel Scholes (1941- ) is a Canadian-born American financial economist.

\(^5\)Max Planck (1858-1947) was a German physicist. He was awarded the Nobel Prize in Physics.
3. The one-dimensional linear transport equation

\[ \partial_t u + a(t, x) \partial_x u = 0 \]

describes transport phenomena such as mass transfer, electric charge transfer and etc.

4. The one-dimensional Kolmogorov\(^6\) or Fokker\(^7\)-Planck equation

\[ \partial_t u - a(t, x) \partial_{xx} u + b(t, x) \partial_x u = 0 \]

gives a statistical description of Brownian\(^8\) motion of a particle in a fluid. The Brownian motion is the seemingly random movement of particles suspended in a fluid, that is a liquid such water or a gas such air.

Practice problems

1. Find all first and second derivatives of each of the functions below

   (a) \( f(x, y) = e^{2x+3y} \),
   
   (b) \( g(x, y) = xy^4 + 2x^3y + 10x - 5y + 20 \),
   
   (c) \( h(x, y) = \sin (4x - 3y) \).

2. Classify each of the following partial differential equations as elliptic, parabolic, or hyperbolic.

   (a) \( \partial_t u + 2\partial_x u = 4\partial_{xx} u \);
   
   (b) \( \partial_{xx} u + 2\partial_{yy} u + \partial_x u + \partial_y u = x + y \);
   
   (c) \( \partial_t u = \partial_{xx} u + b\partial_x u + u \);
   
   (d) \( \partial_t u = c^2\partial_{xx} u - b\partial_x u - ku \) the so-called **telegraph equation**;

3. Show that the function \( u(t, x) = e^{-\alpha^2 t} \sin x \) is a solution of the heat equation

\[ \partial_t u = \alpha^2 \partial_{xx} u. \]

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\(^6\)Andrey Kolmogorov (1903-1987) was a Soviet Russian mathematician, who advanced various scientific fields, among them probability theory, topology, turbulence, classical mechanics, etc.

\(^7\)Adriaan Fokker (1887-1972) was a Dutch physicist and musician.

\(^8\)Robert Brown (1773-1858) was a Scottish botanist.