Lecture XX

Abstract
We introduce parametric description of surfaces in $\mathbb{R}^3$, surface integrals of scalar and vector fields. We also treat oriented surfaces and some applications of Gauss law. We finish this lecture by presenting two fundamental theorems in Vector Calculus having applications in Electromagnetism: the divergence and Stokes theorems.

Parametric surfaces and surface integrals of scalar fields
A parametric surface is a surface in $\mathbb{R}^3$ which is defined by a parametric equation with two parameters. Parametric representation is the most general way to specify a surface. Surfaces that occur in two of the main theorems of vector calculus, Stokes’ theorem and the divergence theorem, which we are going to treat at the end of this lecture are frequently given in a parametric form.

Definition 1 Let

$$ r(t, s) = x(t, s)e_x + y(t, s)e_y + z(t, s)e_z $$

be a vector function defined on a region $D$ of the ts-plane, that is

$$ r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad D = \{(t, s) \in \mathbb{R}^2 \mid a \leq t \leq b, \ c \leq s \leq d\}. $$

A parametric surface $S$ is defined by the point set

$$ S := \{(x, y, z) \in \mathbb{R}^3 \mid x = x(t, s), \ y = y(t, s), \ z = z(t, s) \text{ with } (t, s) \in D\}. $$

Notice that each choice of $(t, s) \in D$ gives a point $(x, y, z)$ on $S$ identified by the vector function $r$.

Example 1 We want to find out the surface $S$ described by the vector function

$$ r(t, s) = 2 \cos t \ e_x + s \ e_y + 2 \sin t \ e_z. $$

According to the definition above points on $S$ with coordinates $x$, $y$ and $z$ will be parameterized according to

$$ x = 2 \cos t, \quad y = s, \quad z = 2 \sin t. $$

By squaring $x$ and adding to it $z^2$ we obtain

$$ x^2 + z^2 = 4 \cos^2 t + 4 \sin^2 t = 4. $$
Thus, we obtain the equation of a circle of radius 2. On the other hand, \( y = s \) tells us that there is no restriction on \( s \) which is allowed to take any real value. We conclude that \( S \) is a cylinder of radius 2 and axis coinciding with the \( y \)-axis.

**Example 2** Find the parametric representation of the surface of a sphere with centre \((0,0,0)\) and radius \(a\)

\[ x^2 + y^2 + z^2 = a^2. \]

The parameterization we need is offered by the spherical coordinates \( r, \vartheta, \) and \( \varphi \) through the relations

\[ x = a \sin \vartheta \cos \varphi, \quad y = a \sin \vartheta \sin \varphi, \quad z = a \cos \vartheta \]

where we set \( r = a \) since we are interested in points lying on the surface of that sphere. Remember that \( \vartheta \in [0, \pi] \) and \( \varphi \in [0, 2\pi] \). Hence, recognizing that the role of the parameters \( t \) and \( s \) are now played by \( \vartheta \) and \( \varphi \), respectively, we conclude that the parametric representation we are looking for is given by

\[ r(\vartheta, \varphi) = a [\sin \vartheta \cos \varphi \, e_x + \sin \vartheta \sin \varphi \, e_y + \cos \vartheta \, e_z]. \]

Surface area is the measure of how much exposed area a solid object has, expressed in square units. Mathematical description of the surface area is considerably more involved than the definition of arc length of a curve. Smooth surfaces, such as a sphere, are assigned surface area using their representation as parametric surfaces. This definition of the surface area is based on methods of infinitesimal calculus and involves partial derivatives and double integration.

**Definition 2** Let \( S \) be a smooth parametric surface (that is \( x, y \) and \( z \) have continuous partial derivatives of any order with respect to \( s \) and \( t \)) given by

\[ r(t, s) = x(t, s)e_x + y(t, s)e_y + z(t, s)e_z, \quad (t, s) \in D. \]

The surface area of \( S \) is

\[ A(S) = \int_D |r_t \times r_s| \, dA, \quad dA = dt ds \quad (1) \]

where

\[ r_t = (\partial_t x) \, e_x + (\partial_t y) \, e_y + (\partial_t z) \, e_z, \quad (2) \]

\[ r_s = (\partial_s x) \, e_x + (\partial_s y) \, e_y + (\partial_s z) \, e_z. \quad (3) \]
Example 3 We want to apply the above formula to compute the surface of a sphere of radius \( a \). In a previous example we already found that the parametrization of the surface of a sphere is

\[
\mathbf{r}(\vartheta, \varphi) = a \left[ \sin \vartheta \cos \varphi \mathbf{e}_x + \sin \vartheta \sin \varphi \mathbf{e}_y + \cos \vartheta \mathbf{e}_z \right].
\]

In order to apply (1) we need

\[
\begin{align*}
\mathbf{r}_\vartheta &= a \left[ \cos \vartheta \cos \varphi \mathbf{e}_x + \cos \vartheta \sin \varphi \mathbf{e}_y - \sin \vartheta \mathbf{e}_z \right], \\
\mathbf{r}_\varphi &= a \left[ -\sin \vartheta \sin \varphi \mathbf{e}_x + \sin \vartheta \cos \varphi \mathbf{e}_y \right],
\end{align*}
\]

which have been computed according to (2) and (3). Moreover,

\[
\mathbf{r}_\vartheta \times \mathbf{r}_\varphi = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a \cos \vartheta \cos \varphi & a \cos \vartheta \sin \varphi & -a \sin \vartheta \\ -a \sin \vartheta \sin \varphi & a \sin \vartheta \cos \varphi & 0 \end{pmatrix} = a^2 \left[ \sin^2 \vartheta \cos \varphi \mathbf{e}_x + \sin^2 \vartheta \sin \varphi \mathbf{e}_y + \sin \vartheta \cos \vartheta \mathbf{e}_z \right].
\]

Finally,

\[
|\mathbf{r}_\vartheta \times \mathbf{r}_\varphi| = a^2 \sin \vartheta.
\]

Applications of (1) gives

\[
A(S) = \iint_D |\mathbf{r}_\vartheta \times \mathbf{r}_\varphi| \, dA = a^2 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta = -2\pi a^2 \cos \vartheta \big|_0^\pi = 4\pi a^2.
\]

Consider a surface \( S \) on which a scalar field \( f \) is defined. If we think of \( S \) as made of some material, and for each point \((x, y, z)\) in \( S \) the number \( f(x, y, z) \) is the density of material at that point, then the surface integral of \( f \) over \( S \) is the mass per unit thickness of \( S \). This only holds as long as the surface is an infinitesimally thin shell. One approach to calculate the surface integral is then to split the surface in many very small pieces, assume that on each piece the density is approximately constant, find the mass per unit thickness of each piece by multiplying the density of the piece by its area, and then sum up the resulting numbers to find the total mass per unit thickness of \( S \).

An explicit formula is presented in the following definition

**Definition 3** Let \( S \) be a surface described by the equation \( z = g(x, y) \) and \( D \) be its projection on the \( xy \)-plane. Then

\[
\iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} \, dA \quad (4)
\]
with $dA = dx\,dy$. In the case that $S$ is given in parametric form by means of the vector function

$$
\mathbf{r}(t, s) = x(t, s)\mathbf{e}_x + y(t, s)\mathbf{e}_y + z(t, s)\mathbf{e}_z, \quad (t, s) \in \tilde{D}
$$

with $a \leq t \leq b$, $c \leq s \leq d$ we have

$$
\iint_S f(x, y, z) \, dS = \iint_{\tilde{D}} f(\mathbf{r}(t, s))|\mathbf{r}_t \times \mathbf{r}_s| \, dA,
$$

(5)

where $\tilde{D} = [a, b] \times [c, d]$ and $dA = dt\,ds$.

**Example 4** Compute the surface integral

$$
\iint_S f(x, y, z) \, dS,
$$

where $f(x, y, z) = y$ and $S$ is the surface $z = x + y^2$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Notice that in the present case $z = g(x, y) = x + y^2$, $\partial_x g = 1$, $\partial_y g = 2y$ and $f(x, y, g(x, y)) = y$ since there is no $z$-dependence in the scalar field $f$. By applying (4) we have

$$
\int\int_S f(x, y, z) \, dS = \int\int_S f(x, y) \, dS = \int\int_D \sqrt{1 + \left(\partial_x g\right)^2 + \left(\partial_y g\right)^2} \, dxdy = \int_0^1 dx \int_0^2 dy \, y\sqrt{2 + 4y^2},
$$

$$
= \sqrt{2} \int_0^2 dy \, y\sqrt{1 + 2y^2}.
$$

By means of the substitution $1 + 2y^2 = \rho$ the above integral becomes

$$
\int\int_S f(x, y, z) \, dS = \sqrt{2} \int_1^9 \frac{d\rho}{4} \sqrt{\rho} = \frac{\sqrt{2}}{4} \int_1^9 d\rho \sqrt{\rho},
$$

$$
= \frac{\sqrt{2}}{4} \frac{2}{3} \rho^{3/2}\big|_1^9 = \frac{\sqrt{2}}{6} (9^{3/2} - 1) = \frac{\sqrt{2}}{6} [(3^2)^{3/2} - 1],
$$

$$
= \frac{\sqrt{2}}{6} (3^3 - 1) = \frac{\sqrt{2}}{6} (27 - 1) = \frac{13 \sqrt{2}}{3}.
$$
Oriented surfaces and surface integrals of vector fields

Imagine that we have a fluid flowing through a surface $S$, such that $v(r)$ determines the velocity of the fluid at $(x, y, z)$. The flux is defined as the quantity of fluid flowing through $S$ in unit amount of time. This illustration implies that if the vector field is tangent to $S$ at each point, then the flux is zero, because the fluid just flows in parallel to $S$, and neither in nor out. This also implies that if $v$ does not just flow along $S$, that is, if $v$ has both a tangential and a normal component, then only the normal component contributes to the flux. Based on this reasoning, to find the flux, we need to take the dot product of $v$ with the unit surface normal to $S$ at each point, which will give us a scalar field, and integrate the obtained field by means of (4) or (5). A unit surface normal to a flat surface is a vector that is perpendicular to that surface. A normal to a non-flat surface at a point $P$ on the surface is a vector perpendicular to the tangent plane to that surface at $P$. A normal to a surface does not have a unique direction: the vector pointing in the opposite direction of a surface normal is also a surface normal.

For a surface which is the boundary of a solid object in three dimensions, one can distinguish between the inward-pointing normal and outer-pointing normal, which can help define the normal in a unique way. In mathematics, orientability is a property of surfaces in Euclidean space measuring whether or not it is possible to make a consistent choice of surface normal vector at every point. In what follows we will always suppose that our surface $S$ is orientable. This requirement rules out the manifestation of pathological surfaces like the Klein bottle or the Möbius strip but this is another story that should be discussed in a Differential Geometry course. Nevertheless, you should be aware that the Möbius strip find applications in electronics. A device called a Möbius resistor is an electronic circuit element which has the property of canceling its own inductive reactance. Nikola Tesla patented similar technology in the early 1900s and was intended for use with his system of global transmission of electricity without wires. To find the unit normal vector $\hat{n}$ to an orientable surface $S$ described by $z = g(x, y)$ or through a parameterization $r(t, s)$ we shall use the formula

$$\hat{n} = \frac{r_t \times r_s}{|r_t \times r_s|} = \frac{-(\partial_x g) e_x - (\partial_y g) e_y + e_z}{\sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2}}.$$  

(6)

**Example 5** Find the unit normal vector $\hat{n}$ for a sphere of radius $a$. We already know that the parametric representation of the surface of a sphere is

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1August Ferdinand Möbius (1790-1868) was a German mathematician and theoretical astronomer.
2Nikola Tesla (1856-1943) was a Serbian inventor, mechanical and electrical engineer.
given by
\[ \mathbf{r}(\vartheta, \varphi) = a[\sin \vartheta \cos \varphi \mathbf{e}_x + \sin \vartheta \sin \varphi \mathbf{e}_y + \cos \vartheta \mathbf{e}_z]. \]

Moreover, from Example 3 we have
\[ \mathbf{r}_\vartheta \times \mathbf{r}_\varphi = a^2 [\sin^2 \vartheta \cos \varphi \mathbf{e}_x + \sin \vartheta \sin \varphi \mathbf{e}_y + \sin \vartheta \cos \varphi \mathbf{e}_z], \]
\[ |\mathbf{r}_\vartheta \times \mathbf{r}_\varphi| = a^2 \sin \vartheta. \]

Therefore,
\[ \hat{n} = \frac{\mathbf{r}_\vartheta \times \mathbf{r}_\varphi}{|\mathbf{r}_\vartheta \times \mathbf{r}_\varphi|} = \sin \vartheta \cos \varphi \mathbf{e}_x + \sin \vartheta \sin \varphi \mathbf{e}_y + \cos \vartheta \mathbf{e}_z = \frac{\mathbf{r}}{a}. \]

Taking into account that \( a = |\mathbf{r}| \) we can also write \( \mathbf{r} = |\mathbf{r}| \hat{n}. \)

**Definition 4** Let \( \mathbf{F} \) be a continuous vector field defined on an oriented surface \( S \) with unit normal vector \( \hat{n} \). We define the surface integral of \( \mathbf{F} \) over \( S \) also called the flux of \( \mathbf{F} \) across \( S \) as follows
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{n} \, dS. \] (7)

**Example 6** Compute the flux of \( \mathbf{F}(x, y, z) = z \mathbf{e}_x + y \mathbf{e}_y + x \mathbf{e}_z \) across the sphere \( x^2 + y^2 + z^2 = 1 \). The parameterization of the sphere is given by \( \mathbf{r}(\vartheta, \varphi) \) as in Example 5. Moreover, employing spherical coordinates we can rewrite the assigned vector field as
\[ \mathbf{F}(\mathbf{r}(\vartheta, \varphi)) = \cos \vartheta \mathbf{e}_x + \sin \vartheta \sin \varphi \mathbf{e}_y + \sin \vartheta \cos \varphi \mathbf{e}_z. \]

According to the previous example the unit normal vector will be
\[ \hat{n} = \mathbf{r}(\vartheta, \varphi). \]

Hence,
\[ \mathbf{F} \cdot \hat{n} = 2 \sin \vartheta \cos \vartheta \cos \varphi \sin^2 \varphi + \sin^2 \vartheta \sin^2 \varphi \]

Applying (7) and (5) we obtain
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{n} \, dS = \iint_{D} (\mathbf{F} \cdot \hat{n}) |\mathbf{r}_\vartheta \times \mathbf{r}_\varphi| \, d\vartheta d\varphi, \]
\[ = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta (2\sin^2 \vartheta \cos \varphi + \sin^3 \vartheta \sin^2 \varphi), \]
\[ = 2 \int_0^{2\pi} d\varphi \cos \varphi \int_0^\pi d\vartheta \sin^2 \vartheta \cos \varphi + \int_0^{2\pi} d\varphi \sin^2 \varphi \int_0^\pi d\vartheta \sin^3 \vartheta. \]
The first integral in the last line is zero since
\[ \int_0^{2\pi} d\phi \cos \phi = \sin(2\pi) - \sin(0) = 0. \]

Therefore, the expression for the flux simplifies to
\[
\iiint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} d\phi \sin^2 \phi \int_0^\pi d\theta \sin^3 \theta.
\]

By means of the trigonometric identity
\[ \sin^2 x = \frac{1 - \cos(2x)}{2} \]
the first integral can be evaluated as follows
\[ \int_0^{2\pi} d\phi \sin^2 \phi = \frac{1}{2} \int_0^{2\pi} d\phi \left[ 1 - \cos(2\phi) \right], \]
\[ = \frac{1}{2} \int_0^{2\pi} d\phi - \frac{1}{2} \int_0^{2\pi} d\phi \cos(2\phi), \]
\[ = \pi - \frac{1}{4} \sin(2\phi) \bigg|_0^{2\pi} = \pi. \]

Finally, we are left with
\[
\iiint_S \mathbf{F} \cdot d\mathbf{S} = \pi \int_0^\pi d\theta \sin^3 \theta,
\]
\[ = \pi \int_0^\pi d\theta \sin \theta \sin^2 \theta,
\]
\[ = \pi \int_0^\pi d\theta \sin \theta (1 - \cos^2 \theta), \]
\[ = \pi \int_0^\pi d\theta \sin \theta - \pi \int_0^\pi d\theta \sin \theta \cos^2 \theta, \]
\[ = -\pi \cos \theta \bigg|_0^\pi + \pi \int_{-1}^1 du u^2, \]
\[ = 2\pi - \pi \int_{-1}^1 du u^2 = 2\pi - \frac{2}{3}\pi = \frac{4}{3}\pi. \]

where we made the substitution \( u = \cos \theta \).

**Example 7** Let \( \mathbf{E} \) be an electric field. Then,
\[ \iiint_S \mathbf{E} \cdot d\mathbf{S} \]
represents the electric flux across $S$. Furthermore, Gauss law says that the net charge inside a closed surface $S$ is

$$Q = \varepsilon_0 \int_S \mathbf{E} \cdot d\mathbf{S},$$

where $\varepsilon_0$ is the electric permittivity of free space. We want to find the total charge $Q$ inside the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$ when the electric field is given by

$$\mathbf{E}(x, y, z) = x \mathbf{e}_x + y \mathbf{e}_y + 2z \mathbf{e}_z.$$

First of all notice, that the surface $S$ across which we want to compute the flux of the electric field consists of a spherical top surface $S_1$ and a lower surface $S_2$ represented by a disk of radius $a$. We choose a positive orientation for $S$, that is outward normal vectors. Thus, the total charge inside $S$ is

$$Q = \varepsilon_0 \int_{S_1} \mathbf{E} \cdot \mathbf{n}_1 dS_1 + \varepsilon_0 \int_{S_2} \mathbf{E} \cdot \mathbf{n}_2 dS_2,$$

Concerning the flux across $S_2$ the outward unit normal vector is simply $\mathbf{n}_2 = -\mathbf{e}_z$ since $S_2$ lies on the $xy$-plane. Further, notice that points on $S_2$ have vanishing $z$ coordinate. Hence,

$$\int_{S_2} (\mathbf{E} \cdot \mathbf{n}_2) dS_2 = -2 \int_{S_2} z dS_2 = -2 \int_{x^2+y^2 \leq a^2, z=0} z dx dy = 0.$$

Thus, the total charge inside $S$ is

$$Q = \varepsilon_0 \int_{S_1} (\mathbf{E} \cdot \mathbf{n}_1) dS_1.$$

From Example 5 we know that

$$\mathbf{n}_1 = \frac{\mathbf{r}}{a} = \frac{x}{a} \mathbf{e}_x + \frac{y}{a} \mathbf{e}_y + \frac{z}{a} \mathbf{e}_z.$$

Moreover,

$$\mathbf{E} \cdot \mathbf{n}_1 = \frac{1}{a} (x^2 + y^2 + 2z^2).$$

Taking into account that the equation of the upper hemisphere is

$$z = g(x, y) = \sqrt{a^2 - (x^2 + y^2)},$$
formula (4) implies

\[ Q = \frac{\varepsilon_0}{a} \iint_{S_1} (x^2 + y^2 + 2z^2) \, dS, \]

\[ = \frac{\varepsilon_0}{a} \iint_{D} \{a^2 + y^2 + 2[a^2 - (x^2 + y^2)]\} \sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} \, dxdy, \]

\[ = \frac{\varepsilon_0}{a} \iint_{D} [2a^2 - (x^2 + y^2)] \sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} \, dxdy, \]

where \( D \) is the projection of the upper hemisphere on the \( xy \)-plane and is given by the disk \( x^2 + y^2 \leq a^2 \). Since

\[ \partial_x g = -\frac{x}{\sqrt{a^2 - (x^2 + y^2)}}, \quad \partial_y g = -\frac{y}{\sqrt{a^2 - (x^2 + y^2)}}, \]

we find that

\[ \sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} = \sqrt{1 + \frac{x^2}{a^2 - (x^2 + y^2)} + \frac{y^2}{a^2 - (x^2 + y^2)}}, \]

\[ = \sqrt{\frac{a^2 - (x^2 + y^2)}{a^2 - (x^2 + y^2)}}, \]

\[ = \frac{a}{\sqrt{a^2 - (x^2 + y^2)}}. \]

Moreover

\[ [2a^2 - (x^2 + y^2)] \sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} = a \frac{2a^2 - (x^2 + y^2)}{\sqrt{a^2 - (x^2 + y^2)}}. \]

Hence, the total charge is given by

\[ Q = \varepsilon_0 \int_{D} \frac{2a^2 - (x^2 + y^2)}{\sqrt{a^2 - (x^2 + y^2)}} \, dxdy. \]

Since \( D \) is the disk \( x^2 + y^2 \leq a^2 \) it results more convenient to use polar coordinates to solve the above integral. To this purpose let \( x = \rho \cos \vartheta \) and \( y = \rho \sin \vartheta \) and recall that the infinitesimal surface element in Cartesian coordinates can be expressed in terms of the infinitesimal surface element in
polar coordinates by means of the determinant of the Jacobian of the trans-
formation \((x, y) \rightarrow (\rho, \vartheta)\) (for more details go back to Lecture XIX), that is
\[ dx dy = \rho \, d\rho d\vartheta. \]

Thus, we obtain
\[ Q = \varepsilon_0 \int_0^{2\pi} d\vartheta \int_0^a d\rho \, \frac{\rho(2a^2 - \rho^2)}{\sqrt{a^2 - \rho^2}} = 2\pi \varepsilon_0 \int_0^a d\rho \, \frac{\rho(2a^2 - \rho^2)}{\sqrt{a^2 - \rho^2}}. \]

Notice that
\[ \frac{d}{d\rho} \sqrt{a^2 - \rho^2} = -\frac{\rho}{\sqrt{a^2 - \rho^2}}. \]

We can rewrite the above integral as
\[ Q = -2\pi \varepsilon_0 \int_0^a d\rho \, (2a^2 - \rho^2) \frac{d}{d\rho} \sqrt{a^2 - \rho^2}. \]

Integration by parts and the substitution \(u = a^2 - \rho^2\) gives
\[ Q = -2\pi \varepsilon_0 \left[ (2a^2 - \rho^2) \sqrt{a^2 - \rho^2} \right]_0^a - \int_0^a d\rho \, (-2\rho) \sqrt{a^2 - \rho^2}, \]
\[ = -2\pi \varepsilon_0 \left[ -2a^3 - \int_0^{a^2} du \, \sqrt{u} \right] = 2\pi \varepsilon_0 \left[ 2a^3 + \int_{a^2}^0 du \, \sqrt{u} \right], \]
\[ = 2\pi \varepsilon_0 \left[ 2a^3 + \frac{2}{3} a^{3/2} \right]_0^a = 2\pi \varepsilon_0 \left[ 2a^3 - \frac{2}{3} a^3 \right] = \frac{8}{3} \pi \varepsilon_0 a^3. \]

**Example 8** Let \(T(x, y, z)\) be the temperature in a solid. The **local heat flow** is defined by the vector field
\[ \mathbf{h} = -\kappa \nabla T, \]
where \(\kappa\) is the thermal conductivity of the solid object. The heat flow across the surface \(S\) enclosing the solid is
\[ \iint_S \mathbf{h} \cdot d\mathbf{S} = -\kappa \iint_S \nabla T \cdot d\mathbf{S}. \]

Suppose that the temperature \(T\) in a metal ball is proportional to the square of the distance from the centre of the ball. Find the heat flux across a sphere of radius \(R\) with centre at the centre of the ball. Introducing a proportionality constant \(C\) the function describing the temperature inside the metal ball can be written as
\[ T(x, y, z) = C(x^2 + y^2 + z^2) = Cr^2. \]
if we set the centre of the ball to be at the origin of the coordinate system.
The local heat flux is
\[ h = -\kappa C \nabla (x^2 + y^2 + z^2) = -2\kappa C (x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z) = -2\kappa C \mathbf{r}. \]

The outward unit normal to the sphere \( x^2 + y^2 + z^2 = R^2 \) is
\[ \mathbf{n} = \frac{\mathbf{r}}{R}. \]

Hence,
\[ \iint_S h \cdot dS = \iint_S (h \cdot \mathbf{n}) \ dS. \]

In particular,
\[ h \cdot \mathbf{n} = -2\kappa C \frac{\mathbf{r} \cdot \mathbf{r}}{R} = -2\kappa C \frac{r^2}{R}. \]

The flux across \( S \) becomes
\[ \iint_S h \cdot dS = \iint_S (h \cdot \mathbf{n}) \ dS = -\frac{2\kappa C}{R} \iint_S r^2 \ dS. \]

Notice that on the surface \( S \) of our sphere of radius \( R \) we have \( r = R \). Thus, the above integral can be computed as follows
\[ \iint_S h \cdot dS = -\frac{2\kappa C}{R} R^2 \iint_S dS = -2\kappa C R \cdot 4\pi R^2 = -8\pi \kappa C R^3. \]

Stokes Theorem relates a surface integral over a surface \( S \) to a line integral around the boundary curve of \( S \).

**Theorem 1 (Stokes theorem)**

Let \( S \) be a positively oriented surface \( S \) bounded by a simple, closed boundary curve \( C \) with positive orientation. Let \( \mathbf{F} \) be a vector field whose components have continuous partial derivatives on an open region in \( \mathbb{R}^3 \) containing \( S \). Then
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}. \]  

(8)

Notice that if the surface \( S \) lies on the \( xy \)-plane with upward orientation (\( \mathbf{n} = \mathbf{e}_z \)), then Stokes theorem reduces to Green’s theorem since
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \ dS = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{e}_z \ dS = \int_S (\nabla \times \mathbf{F})_z \ dS
= \int_S (\partial_y F_x - \partial_x F_y) \ dS.
\]
Example 9  Consider the vector field
\[ \mathbf{F}(x, y, z) = yz \mathbf{e}_x + xz \mathbf{e}_y + xy \mathbf{e}_z. \]

Compute
\[ \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}, \]
where \( S \) is the upper intersection of the sphere \( x^2 + y^2 + z^2 = 4 \) with the cylinder \( x^2 + y^2 = 1 \). There are two ways of solving this problem. The first one is the shortest and relies on the observation that the given vector field is conservative since a straightforward computation shows that \( \nabla \times \mathbf{F} = 0 \), thus implying that the flux of \( \nabla \times \mathbf{F} \) through \( S \) is zero. The second method is longer and makes use of the Stokes theorem to transform the given surface integral into a line integral along the boundary of \( S \), that is
\[ \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}. \quad (9) \]

The curve \( C \) is the boundary of a disk perpendicular to the \( z \)-axis, parallel to the \( xy \)-plane and having a certain height \( z \) with respect to the \( xy \)-plane. The points \( (x, y, z) \) of the intersection of the two given solid objects can be found by solving the system of equations
\[ x^2 + y^2 + z^2 = 4, \quad x^2 + y^2 = 1 \]
from which we obtain
\[ 1 + z^2 = 4 \quad \Rightarrow \quad z = \pm \sqrt{3}. \]

Since we are considering the upper intersection we have to choose \( z = \sqrt{3} \). Hence, point lying on the closed curve \( C \) are characterized as follows
\[ x^2 + y^2 = 1, \quad z = \sqrt{3}. \quad (10) \]

To compute the line integral in (9) we need a parametrization of \( C \) in terms of a vector function \( \mathbf{r}(t) \). From (10) we see that the \( x \) and \( y \) coordinates of points on \( C \) satisfy the equation of a circle of unit radius for which a natural parameterization is given by
\[ x(t) = \cos t, \quad y(t) = \sin t, \quad t \in [0, 2\pi]. \]
Thus, the parameterization of \( C \) is given by the following vector
\[ \mathbf{r}(t) = \cos t \mathbf{e}_x + \sin t \mathbf{e}_y + \sqrt{3} \mathbf{e}_z. \]
and the vector field $\mathbf{F}$ along $C$ will be

$$\mathbf{F}(x, y, z) = yz \mathbf{e}_x + xz \mathbf{e}_y + xy \mathbf{e}_z = \sqrt{3} \sin t \mathbf{e}_x + \sqrt{3} \cos t \mathbf{e}_y + \sin t \cos t \mathbf{e}_z.$$  

The line integral (9) can now be computed by the formula for line integrals of vector fields and we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt},$$

$$= \int_0^{2\pi} dt \left[ \sqrt{3} \sin t \mathbf{e}_x + \sqrt{3} \cos t \mathbf{e}_y + \sin t \cos t \mathbf{e}_z \right] \cdot \left[ -\sin t \mathbf{e}_x + \cos t \mathbf{e}_y \right],$$

$$= \sqrt{3} \int_0^{2\pi} dt \left( \cos^2 t - \sin^2 t \right) = \sqrt{3} \int_0^{2\pi} dt \cos (2t),$$

$$= \sqrt{3} \sin (2t) \bigg|_0^{2\pi} = 0.$$

The next theorem relates surface integrals with volume integrals.

**Theorem 2 (Divergence theorem)**

Let $E$ be a solid region and $S$ the boundary surface of $E$ having positive orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing $E$. Then,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV.$$  

(11)

**Example 10** We want to apply Gauss

$$Q = \epsilon_0 \int_S \mathbf{E} \cdot d\mathbf{S},$$

to compute the net charge inside a closed surface $S$ of the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$ when the electric field is given by

$$\mathbf{E}(x, y, z) = x \mathbf{e}_x + y \mathbf{e}_y + 2z \mathbf{e}_z.$$  

First of all notice that due to the symmetry of the problem the charge inside the given solid hemisphere will be one half of the total charge inside the sphere $x^2 + y^2 + z^2 = a^2$. If $\Sigma$ denote the surface of that sphere we have

$$Q = \epsilon_0 \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{\epsilon_0}{2} \iint_\Sigma \mathbf{E} \cdot d\Sigma.$$  

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and applying the divergence theorem we find

$$Q = \frac{\epsilon_0}{2} \iiint_E (\nabla \cdot E) \, dV,$$

where $E$ denotes the solid region whose boundary is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. A trivial computation shows that

$$\nabla \cdot E = \partial_x x + \partial_y y + \partial_z (2z) = 4.$$

Hence, the total charge inside the given hemisphere is

$$Q = \frac{\epsilon_0}{2} \iiint_E (\nabla \cdot E) \, dV = 2\epsilon_0 \int \int dV = 2\epsilon_0 \cdot \frac{4}{3} \pi a^3 = \frac{8}{3} \pi \epsilon_0 a^3.$$

**Example 11** Let $a$ be a constant vector field. Shows that

$$\int_S a \cdot dS = 0.$$

This is a trivial consequence of the divergence theorem. Notice that since each component of the given vector field is constant, we have $\nabla \cdot a = 0$.

**Example 12** Let $\phi$ and $\psi$ be scalar fields. Prove Green’s identity

$$\int_S (\phi \nabla \psi) \cdot dS = \int \int \int_E [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] \, dV.$$

We have to apply the divergence theorem for a vector field $F = \phi \nabla \psi$ which gives

$$\int_S (\phi \nabla \psi) \cdot dS = \int \int \int_E \nabla (\phi \nabla \psi) \, dV.$$

Now notice that

$$\nabla (\phi \nabla \psi) = (\partial_x e_x + \partial_y e_y + \partial_z e_z) \cdot (\phi \partial_x e_x + \phi \partial_y e_y + \phi \partial_z e_z),$$

$$= \partial_x (\phi \partial_x \psi) + \partial_y (\phi \partial_y \psi) + \partial_z (\phi \partial_z \psi),$$

$$= \partial_x \phi \partial_x \psi + \phi \partial_{xx} \psi + \partial_y \phi \partial_y \psi + \partial_{yy} \psi + \partial_z \phi \partial_z \psi + \phi \partial_{zz} \psi,$$

$$= \phi \partial_{xx} \psi + \phi \partial_{yy} \psi + \phi \partial_{zz} \psi + \partial_x \phi \partial_x \psi + \partial_y \phi \partial_y \psi + \partial_z \phi \partial_z \psi,$$

$$= \phi (\partial_{xx} \psi + \partial_{yy} \psi + \partial_{zz} \psi) +$$

$$+(\partial_x \phi e_x + \partial_y \phi e_y + \partial_z \phi e_z) \cdot (\partial_x e_x + \partial_y e_y + \partial_z e_z)$$

$$= \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi).$$
Practice problems

1. Prove the following identity

\[ \nabla \times (\nabla \times F) = \text{grad} (\text{div} F) - \nabla^2 F \]

assuming that the appropriate partial derivatives exist and are continuous.

2. If

\[ F = \frac{r}{r^p} \]

find \( \nabla F \). Is there a value of \( p \) for which \( \nabla F = 0 \)?

3. This exercise shows a connection between the curl vector and rotations. Let \( B \) be a rigid body (for instance a disk) rotating about the \( z \)-axis. The rotation can be described by the vector \( \mathbf{w} = \omega \mathbf{e}_z \), where \( \omega \) is the angular velocity of \( B \), that is, the tangential speed of any point \( P \) in \( B \) divided by the distance \( d \) from the axis of rotation. Let \( \mathbf{r} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z \) be the position vector of \( P \).

   (a) By considering the angle \( \vartheta \) giving the inclination of the vector \( \mathbf{r} \) identifying the point \( P \) show that the velocity field of \( B \) is \( \mathbf{v} = \mathbf{w} \times \mathbf{r} \).

   (b) Show that \( \mathbf{v} = -\omega y \mathbf{e}_x + \omega x \mathbf{e}_y \).

   (c) Show that \( \nabla \times \mathbf{v} = 2\mathbf{w} \).

4. Maxwell equations relating the electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \) as they vary with time in a region containing no charge and no current can be stated as follows

\[
\begin{align*}
\text{div} \mathbf{E} &= 0, \\
\text{div} \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{1}{c} \partial_t \mathbf{B}, \\
\nabla \times \mathbf{B} &= \frac{1}{c} \partial_t \mathbf{E},
\end{align*}
\]

where \( c \) is the speed of light in vacuum. Use these equations to prove the following

   (a) \( \nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \partial_{tt} \mathbf{E} \)

   (b) \( \nabla \times (\nabla \times \mathbf{B}) = -\frac{1}{c^2} \partial_{tt} \mathbf{B} \)

   (c) \( \nabla^2 \mathbf{E} = \frac{1}{c^2} \partial_{tt} \mathbf{E} \) (Hint: Use problem 1.)
(d) $\nabla^2 B = \frac{1}{c^2} \partial_t B.$

5. Identify the surface with the following vector equation
\[ \mathbf{r}(x, \vartheta) = x \mathbf{e}_x + \cos \vartheta \mathbf{e}_y + \sin \vartheta \mathbf{e}_z. \]

6. Find a parametric representation for the plane passing through the point $(1, 2, -3)$ and containing the vectors $\mathbf{e}_x + \mathbf{e}_y - \mathbf{e}_z$ and $\mathbf{e}_x - \mathbf{e}_y + \mathbf{e}_z$.

7. Find a parametric representation for the part of the plane $z = 5$ that lies inside the cylinder $x^2 + y^2 = 16$.

8. Find the area of the helicoid (or spiral ramp) with vector equation
\[ \mathbf{r}(u, v) = u \cos v \mathbf{e}_x + u \sin v \mathbf{e}_y + v \mathbf{e}_z \]
with $0 \leq u \leq 1$ and $0 \leq v \leq \pi$.

9. Evaluate the surface integral
\[ \iint_S x^2yz \, dS, \]
where $S$ is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$.

10. Evaluate the surface integral
\[ \iint_S yz \, dS, \]
where $S$ is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

11. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface $S$. In other words, find the flux of $\mathbf{F}$ across $S$. For closed surfaces, use the positive (outward) orientation.

(a) $\mathbf{F}(x, y, z) = xy \mathbf{e}_x + 4x^2 \mathbf{e}_y + yz \mathbf{e}_z$, $S$ is the surface $z = xe^y$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with upward orientation.

(b) $\mathbf{F}(x, y, z) = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z$, $S$ is the sphere $x^2 + y^2 + z^2 = 9$.

(c) $\mathbf{F}(x, y, z) = -y \mathbf{e}_x + x \mathbf{e}_y + 3z \mathbf{e}_z$, $S$ is the hemisphere $z = \sqrt{16 - (x^2 + y^2)}$ with upward orientation.
12. Use Gauss law to find the charge enclosed by the cube with vertices \((\pm 1, \pm 1, \pm 1)\) if the electric field is
\[
E(x, y, z) = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z.
\]

13. The temperature at the point \((x, y, z)\) in a substance with conductivity \(\kappa = 6.5\) is
\[
T(x, y, z) = 2y^2 + 2z^2.
\]
Find the rate of heat flow inward across the cylindrical surface \(y^2 + z^2 = 6, 0 \leq x \leq 4\).

14. The temperature at a point in a ball with conductivity \(\kappa\) is inversely proportional to the distance from the centre of the ball. Find the rate of heat flow across a sphere \(S\) of radius \(a\) with centre at the centre of the ball.

15. Use Stokes theorem to evaluate \(\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\) for
\[
\begin{align*}
(a) & \quad \mathbf{F}(x, y, z) = x^2e^{yz} \mathbf{e}_x + y^2e^{xz} \mathbf{e}_y + z^2e^{xy} \mathbf{e}_z \text{ and } S \text{ is the hemisphere } x^2 + y^2 + z^2 = 4, z \geq 0 \text{ oriented upward.} \\
(b) & \quad \mathbf{F}(x, y, z) = [x + \arctan (yz)] \mathbf{e}_x + y^2z \mathbf{e}_y + z \mathbf{e}_z \text{ and } S \text{ is the part of the hemisphere } x = \sqrt{9 - x^2 - y^2} \text{ that lies inside the cylinder } y^2 + z^2 = 4, \text{ oriented in the direction of the positive } x\text{-axis.}
\end{align*}
\]

16. Use Stokes theorem to evaluate \(\oint_C \mathbf{F} \cdot d\mathbf{r}\) for
\[
\begin{align*}
(a) & \quad \mathbf{F}(x, y, z) = (x + y^2) \mathbf{e}_x + (y + z^2) \mathbf{e}_y + (z + x^2) \mathbf{e}_z \text{ and } C \text{ is the triangle with vertices } (1, 0, 0), (0, 1, 0) \text{ and } (0, 0, 1). \\
(b) & \quad \mathbf{F}(x, y, z) = 2z \mathbf{e}_x + 4x \mathbf{e}_y + 5y \mathbf{e}_z \text{ and } C \text{ is the curve of intersection of the plane } z = x + 4 \text{ and the cylinder } x^2 + y^2 = 4.
\end{align*}
\]

17. Calculate the work done by the force field
\[
\mathbf{F}(x, y, z) = (x^2 + z^2) \mathbf{e}_x + (y^2 + x^2) \mathbf{e}_y + (z^2 + y^2) \mathbf{e}_z
\]
when a particle moves under its influence around the edge of the part of the sphere \(x^2 + y^2 + z^2 = 4\) that lies in the first octant, in a counterclockwise direction as viewed from above.

18. Use the divergence theorem to evaluate the flux of \(\mathbf{F}\) across \(S\) for
\[
\begin{align*}
(a) & \quad \mathbf{F}(x, y, z) = 3y^2z^3 \mathbf{e}_x + 9x^2yz^2 \mathbf{e}_y - 4xy^2 \mathbf{e}_z \text{ and } S \text{ is the surface of the cube with vertices } (\pm 1, \pm 1, \pm 1).
\end{align*}
\]
(b) \( \mathbf{F}(x, y, z) = -xz \, \mathbf{e}_x - yz \, \mathbf{e}_y + z^2 \, \mathbf{e}_z \) and \( S \) is the ellipsoid
\[
\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1.
\]

19. Verify that \( \nabla \cdot \mathbf{E} = 0 \) for the electric field
\[
\mathbf{E}(x, y, z) = \epsilon \frac{Q}{|\mathbf{r}|^3} \, \mathbf{x}.
\]