Lecture XIII

Abstract

We introduce the Laplace equation in cylindrical coordinates and apply the method of separation of variables to solve it. We show that in the case of azimuthal symmetry the radial equation reduces to the well-known Bessel equation of order zero. Finally, we treat an application in electrostatics concerning the computation of the electric potential inside a cylindrical can whose top is held at a certain given potential whereas the lateral surface and bottom are kept insulated.

The Laplace equation in cylindrical coordinates

Solutions to the Laplace equation in cylindrical coordinates have wide applicability from fluid mechanics to electrostatics. To convert the Laplace equation in Cartesian coordinates

\[ 0 = \nabla^2 \Phi = \partial_{xx} \Phi + \partial_{yy} \Phi + \partial_{zz} \Phi = 0, \quad \Phi = \Phi(x, y, z), \]

let us introduce cylindrical coordinates defined through the relations

\[ x = r \cos \vartheta, \quad y = r \sin \vartheta, \quad z = z \]

where \( r \geq 0, \vartheta \in [0, 2\pi) \) and \( z \in (-\infty, \infty) \). Here, \( r \) is the polar distance of a point \( P \) with coordinates \((x, y, z)\) from \( z \)-axis, \( \vartheta \) is the azimuthal angle between the reference direction on the \( xy \)-plane and the line from the origin to the projection of \( P \) on that plane and \( z \) is the height, that is the distance from the \( xy \)-plane plane to the point \( P \). You can find more information about cylindrical coordinates and some pictorial representations at the following links

- http://www.math.uic.edu/classes/eecs/eecs520/textbook/node31.html

The inverse formulas expressing cylindrical coordinates in terms of the Cartesian coordinates are

\[ r = \sqrt{x^2 + y^2}, \quad \vartheta = \arctan \left( \frac{y}{x} \right). \]

To derive the Laplace equation in cylindrical coordinates we have to express \( \partial_{xx} \Phi \) and \( \partial_{yy} \Phi \) in terms of partial derivatives with respect to \( r \) and \( \vartheta \) by means of the chain rule (see Problem 1 in the Problem Section in Lecture XII). Doing that we find

\[ 0 = \nabla^2 \Phi = \partial_r \Phi + \frac{1}{r} \partial_r \Phi + \frac{1}{r^2} \partial_{\vartheta \vartheta} \Phi + \partial_{zz} \Phi = 0, \quad \Phi = \Phi(r, \vartheta, z). \]
Separation of the Laplace equation in cylindrical coordinates

Let us introduce the guess

$$\Phi(r, \vartheta, z) = R(r)A(\vartheta)S(z).$$  \hfill (3)

Substitution into (2) gives

$$A(\vartheta)S(z)\frac{d^2R}{dr^2} + \frac{A(\vartheta)S(z)}{r} \frac{dR}{dr} + \frac{R(r)S(z)}{r^2} \frac{d^2A}{d\vartheta^2} + R(r)A(\vartheta)\frac{d^2S}{dz^2} = 0.$$ 

Dividing the above expression by $R(r)A(\vartheta)S(z)$ we obtain

$$\frac{1}{R(r)} \frac{d^2R}{dr^2} + \frac{1}{rR(r)} \frac{dR}{dr} + \frac{1}{r^2A(\vartheta)} \frac{d^2A}{d\vartheta^2} + \frac{1}{S(z)} \frac{d^2S}{dz^2} = 0.$$ 

Rewrite the previous expression as follows

$$\frac{1}{R(r)} \frac{d^2R}{dr^2} + \frac{1}{rR(r)} \frac{dR}{dr} + \frac{1}{r^2A(\vartheta)} \frac{d^2A}{d\vartheta^2} = -\frac{1}{S(z)} \frac{d^2S}{dz^2}.$$ 

Since the l.h.s. depends on $r$ and $\vartheta$ whereas the r.h.s. on $z$ only, equality will hold if

$$\frac{1}{R(r)} \frac{d^2R}{dr^2} + \frac{1}{rR(r)} \frac{dR}{dr} + \frac{1}{r^2A(\vartheta)} \frac{d^2A}{d\vartheta^2} = -\lambda = -\frac{1}{S(z)} \frac{d^2S}{dz^2},$$

where $\lambda$ is a separation constant. Therefore, we end up with the equations

$$\frac{1}{R(r)} \frac{d^2R}{dr^2} + \frac{1}{rR(r)} \frac{dR}{dr} + \frac{1}{r^2A(\vartheta)} \frac{d^2A}{d\vartheta^2} = -\lambda,$$  \hfill (4)

$$\frac{d^2S}{dz^2} - \lambda S(z) = 0.$$  \hfill (5)

Equation (4) can be further separated by rewriting it as follows

$$\frac{1}{R(r)} \frac{d^2R}{dr^2} + \frac{1}{rR(r)} \frac{dR}{dr} + \lambda = -\frac{1}{r^2A(\vartheta)} \frac{d^2A}{d\vartheta^2}.$$ 

If we multiply the above expression by $r^2$ we end up with the equation

$$\frac{r^2}{R(r)} \frac{d^2R}{dr^2} + \frac{r}{R(r)} \frac{dR}{dr} + \lambda r^2 = -\frac{1}{A(\vartheta)} \frac{d^2A}{d\vartheta^2},$$

where the l.h.s. of the above equation depends on $r$ only and the r.h.s. is a function of $\vartheta$. Equality will hold if

$$\frac{r^2}{R(r)} \frac{d^2R}{dr^2} + \frac{r}{R(r)} \frac{dR}{dr} + \lambda r^2 = -\mu = -\frac{1}{A(\vartheta)} \frac{d^2A}{d\vartheta^2}.$$
Thus, we obtain two more ordinary differential equations, namely
\[
\frac{r^2}{R(r)} \frac{d^2 R}{dr^2} + \frac{r}{R(r)} \frac{dR}{dr} + \lambda r^2 = -\mu,
\]
\[
\frac{d^2 A}{d\vartheta^2} - \mu A(\vartheta) = 0.
\]
We have thus obtained three ordinary differential equations that we are going to rewrite as follows
\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left( \lambda r + \frac{\mu}{r} \right) R(r) = 0, \quad (6)
\]
\[
\frac{d^2 A}{d\vartheta^2} - \mu A(\vartheta) = 0, \quad (7)
\]
\[
\frac{d^2 S}{dz^2} - \lambda S(z) = 0. \quad (8)
\]

**The case of azimuthal symmetry**

In the case of azimuthal symmetry the potential Φ does not depend on the angular variable \( \vartheta \). Remember that a physical system will be invariant under rotations of an angle \( \vartheta \) about the \( z \)-axis if its configuration does not change. If we would apply the method of separation of variables in the present case we would start with a guess
\[
\Phi(r, \vartheta) = R(r)S(z). \quad (9)
\]
If we compare (9) with (3) we discover that \( A(\vartheta) \) must be one. Since \( A \) is now a constant equation (7) requires that \( \mu = 0 \). Thus, we are left with the problem of solving the equations
\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right) + \lambda r R(r) = 0, \quad (10)
\]
\[
\frac{d^2 S}{dz^2} - \lambda S(z) = 0. \quad (11)
\]
To find the general solution of (10) we rewrite the equation as
\[
r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \lambda r R(r) = 0 \quad (13)
\]
and make the change of variable
\[
r = \frac{t}{\sqrt{\lambda}}, \quad \lambda > 0.
\]
Then, we have
\[
\frac{dR}{dr} = \frac{dR}{dt} \frac{dt}{dr} = \sqrt{\lambda} \frac{dR}{dt}, \quad \frac{d^2 R}{dr^2} = \lambda \frac{d^2 R}{dt^2}
\]
and (13) becomes
\[
\frac{t}{\sqrt{\lambda}} \cdot \lambda \frac{d^2 R}{dt^2} + \sqrt{\lambda} \frac{dR}{dt} + \lambda \cdot \frac{t}{\sqrt{\lambda}} R(t) = 0,
\]
which simplifies to
\[
t \frac{d^2 R}{dt^2} + \frac{dR}{dt} + tR(t) = 0. \tag{14}
\]
We succeeded in reducing the radial equation to the Bessel equation of order zero. As we already know from Lecture V the general solution of the above equation is
\[
R(t) = c_1 J_0(t) + c_2 Y_0(t).
\]
Transforming back to \( r \) we get the general solution of the radial equation
\[
R(r) = c_1 J_0(\sqrt{\lambda}r) + c_2 Y_0(\sqrt{\lambda}r). \tag{15}
\]
Concerning the solution of the equation governing \( S(z) \) since \( \lambda > 0 \) we find
\[
S(z) = Ae^{\sqrt{\lambda}z} + Be^{-\sqrt{\lambda}z}.
\]
The form of the solution of the Laplace equation in cylindrical coordinates depends on the boundary conditions imposed in a particular problem arising in physics or engineering. Instead of presenting an abstract treatment of the general solution we switch to an application from electrostatics. This example should clarify how the solution can be derived starting from the previous considerations and certain assigned boundary conditions.

**An example from electrostatics**

We consider a conducting cylindrical can whose top has a given potential \( V(r) \) with the lateral surface and the bottom grounded. Let \( a \) be the radius of the can and \( h \) its height. We want to find the electrostatic potential \( \Phi \) at all points inside the can.

First of all, notice that this problem exhibits azimuthal symmetry. Hence, the electrostatic potential will depend on \( r \) and \( z \) only. Hence, we can apply all relevant information we derived in the previous section. However, in order to solve the angular equation and to find the eigenvalues for the Sturm-Liouville problem associated to the radial equation we need to impose certain boundary conditions reflecting the problem we are solving. Such boundary conditions are the following.
1. The potential must vanish asymptotically at the bottom of the can

\[ \Phi(r, 0) = 0. \]

Since \( \Phi(r, z) = R(r)S(z) \) the above condition implies \( S(0) = 0 \). Recalling that in the previous section we found

\[ S(z) = Ae^{\sqrt{\lambda}z} + Be^{-\sqrt{\lambda}z}, \]

we conclude that \( S(0) = 0 \) implies \( A + B = 0 \), that is \( B = -A \). Hence,

\[ S(z) = A \left( e^{\sqrt{\lambda}z} - e^{-\sqrt{\lambda}z} \right) = 2A \sinh(\sqrt{\lambda}z). \]

2. The potential \( \Phi \) must be finite along the central axis of the can, that is

\[ \Phi(0, z) < \infty \quad \text{for} \quad r = 0 \quad \text{and} \quad 0 < z < h. \]

Having in mind that the Bessel function \( Y_0(\sqrt{\lambda}r) \) entering in the general solution of the radial equation diverges for \( r = 0 \), the above condition will be satisfied if \( C_2 = 0 \) in (15). Therefore, the radial function \( R \) simplifies to

\[ R(r) = c_1 J_0(\sqrt{\lambda}r). \]

3. On the lateral surface of the can the potential vanishes, that is

\[ \Phi(a, z) = 0 \]

This implies that \( R(a) = 0 \). For \( c_1 \neq 0 \) this condition implies

\[ J_0(\sqrt{\lambda}a) = 0. \quad (16) \]

If you give a look at the graph of \( J_0 \) I draw in class for the Lecture V you will realize that this Bessel function intersects the positive \( x \)-axis an infinite number of times. This implies that (16) has infinitely many roots which we denote by \( x_n \) with \( n = 1, 2, \cdots \). Hence, the Sturm-Liouville problem associated to the radial equation admits a non-trivial solution whenever

\[ \sqrt{\lambda_n}a = x_n \implies \sqrt{\lambda_n} = \frac{x_n}{a}, \quad n = 1, 2, \cdots. \]

Again from the Sturm-Liouville theory we know that the set

\[ \{ J_0(x_1r/a), J_0(x_2r/a), \cdots, J_0(x_nr/a), \cdots \} \]
is an orthogonal set and since the Laplace equation is linear we can write the solution of the potential governing our problem as

\[ \Phi(r, z) = \sum_{n=1}^{\infty} c_n J_0 \left( \frac{x_n}{a} r \right) \sinh \left( \frac{x_n}{a} z \right). \]  

(17)

4. The last boundary condition

\[ \Phi(r, h) = V(r) \]  

(18)

together with the orthogonality property of the Bessel functions play a fundamental role in finding the constants \( c_n \) in the expansion (17). Remember that \( V(r) \) is an assigned function describing the electrostatic potential on the top of the can. By means of (18) equation (17) becomes

\[ \Phi(r, h) = \sum_{n=1}^{\infty} c_n J_0 \left( \frac{x_n}{a} r \right) \sinh \left( \frac{x_n}{a} h \right). \]  

(19)

Taking into account the following orthogonality relation between Bessel functions of order zero

\[ \int_0^a dr \, r J_0 \left( \frac{x_n}{a} r \right) J_0 \left( \frac{x_m}{a} r \right) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{a^2}{2} J_1(x_n) & \text{if } m = n, \end{cases} \]  

(20)

where \( x_n \) is the \( n \)-th root of \( J_0(x) \), we multiply (19) by \( r J_0(x_m r/a) \) and integrate over the interval \([0, a]\) to get

\[ \int_0^a dr \, r V(r) J_0 \left( \frac{x_m}{a} r \right) = \sum_{n=1}^{\infty} c_n \sinh \left( \frac{x_n}{a} h \right) \int_0^a dr \, J_0 \left( \frac{x_n}{a} r \right) J_0 \left( \frac{x_m}{a} r \right). \]

Applying (20) we have

\[ \int_0^a dr \, r V(r) J_0 \left( \frac{x_m}{a} r \right) = \frac{a^2 c_m}{2} \sinh \left( \frac{x_m}{a} h \right) J_1(x_m). \]

Finally, we find

\[ c_m = \frac{2}{a^2 J_1(x_m) \sinh \left( \frac{x_m}{a} h \right)} \int_0^a dr \, r V(r) J_0 \left( \frac{x_m}{a} r \right), \quad m = 1, 2, \cdots. \]
Practice problems

1. Suppose that the top face of a conducting cylinder is held at the constant potential $V_0$ while the lateral surface and the bottom face are grounded. Find the electrostatic potential $\Phi$ at all points inside the can.

2. A long conducting cylinder of radius $a$ is composed of two halves (with semicircular cross sections) with an infinitesimal gap between them. The upper and lower halves of the cylinder are in contact with heat baths of temperatures $T_0$ and $-T_0$, respectively. Find the temperature both inside and outside the cylinder. **Hint:** assume that the temperature distribution $T$ satisfies the Laplace equation.

3. A long conducting cylinder of radius $a$ is composed of two halves (with semicircular cross sections) with an infinitesimal gap between them. The upper and lower halves of the cylinder are in contact with heat baths of temperatures $T_1$ and $-T_1$, respectively. The cylinder is inside a larger cylinder (and coaxial with it) held at temperature $T_2$. Find the temperature inside the inner cylinder, between the two cylinders and outside the outer cylinder.

4. A long conducting cylinder of radius $a$ is kept at potential $V_1$. The cylinder is inside a larger cylinder (and coaxial with it) held at potential $V_2$. Find the potential inside the inner cylinder, between the two cylinders, and outside the outer cylinder.