1. Introduction

General Relativity and Black Holes

In General Relativity, the gravitational field of a massive object can be thought of as the curvature of a four-dimensional space-time and it is described by a metric tensor \( g_{\mu\nu} \) and the Riemann tensor \( R_{\mu\nu\rho\sigma} \). The Einstein field equations provide a connection between the space-time geometry and the matter distribution:

\[
G_{\mu\nu} = 8\pi T_{\mu\nu}
\]

where \( G_{\mu\nu} \) is the Ricci tensor, \( T_{\mu\nu} \) is the stress-energy tensor, and \( 8\pi \) is a proportionality constant. The field equations predict the behavior of massive objects and their gravitational effects around them. A stationary, axially symmetric, vacuum solution of the Einstein field equations is the Schwarzschild solution, in which the mass concentration is so dense that even light would be trapped. The horizon of a black hole marks the point of no return, beyond which the gravitational force is so strong that not even light can escape. The event horizon, on the other hand, is the surface where the gravitational force is strong enough to prevent the escape of any material or information. Beyond the event horizon, the black hole's gravity is so strong that even time itself becomes distorted, leading to the phenomenon of time dilation.

In Boyer-Lindquist coordinates (\( t, r, \theta, \phi \)), a solution is given by the line element

\[
ds^2 = -c^2 dt^2 + r^2 d\Omega^2 + \left(1 - \frac{2M}{r}\right) dr^2 + 2r dtdr
\]

where \( c \) is the speed of light, \( M \) is the mass of the central object, and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). This solution describes the spacetime surrounding a non-rotating black hole, known as the Schwarzschild black hole.

3. Next goals

- In the theory of special functions the generalised Heun equation is very important since it contains the elliptic Legendre equation as well as the Heun equation as special cases. In [4] we studied the separated radial part of a massive Dirac equation in the Kerr-Newman metric and the static perturbations for the non-rotating, self-balanced, Newether solution. In [4] we begin with the generalised Heun equation modified by a real parameter \( \nu \), which is a key point in our study of the gravitational field of a spinning mass as an example of algebro-geometric type equation. I s it possible to do the same for the generalised spheroidal equation or other eigenvalue problems? If the answer is positive, this new approach would represent an alternative methodology for obtaining a class of eigenvalue problems. This is a joint work with Daniel Chan (UWI) and Mark Nowakowski. (Universidad de Los Andes).

- Quantum Gravity has not yet converged into a single theory and at present several rival theories co-exist (String Theory, Loop Quantum Gravity and Noncommutative Geometry). Nevertheless, all these candidates reveal certain conceptual or global features like noncommutativity of the coordinates at Planck scales. This quantum aspect of space-time has been recently applied to the final state of a black hole [13]. The starting point of these new developments is the commutation relation between coordinate operators. Based on such a commutation relation one can show that the amplitude (\( \psi \)) which in the standard quantum mechanical prescription is proportional to the Dirac delta function, becomes now a Gaussian distribution. Therefore, one of the effects of the quantum gravity is the replacement of a sharp distance by a Gaussian smearing. This observation gave rise to new models of black holes [12] where the final singularity is replaced by a self-gravitating droplet. Although the zone of smeared distance might not be the physical quantum geometry, these models explicitly reveal its importance. For instance, at Planck scales where the curvature is of the order of the Planck scale and not of the Planck mass, there is no geometric significance in the geometry of the universe.