The 2010 Jamaican Mathematical Olympiad
Presented by
The University of the West Indies
in Collaboration with
Sterling Asset Management Ltd.

First Round Examination
Solutions for Grades 9, 10 and 11

Part A

1) First, 40% of 400 is \(0.40 \times 400 = 160\). Then 30% of 40% of 400 is 30% of 160. This is \(0.30 \times 160 = 48\).

2) Let \(m\) be the amount of money Mrs. Brown started with. After she spent \(\frac{1}{3}m\) on her groceries, she had \(\frac{2}{3}m\) remaining. Since she then deposited $1,000 and had $200 left over, this means that
\[
\frac{2}{3}m - 1000 = 200; \quad \frac{2}{3}m = 1,200; \quad m = \frac{3}{2}(1,200) = \frac{3,600}{2} = 1,800.
\]
Mrs. Brown started with $1,800.

3) First, \(\angle AOB + \angle BOC + \angle COD = 180^\circ\). Then \(\angle AOB + 60^\circ + \angle COD = 180^\circ\) and so \(\angle AOB + \angle COD = 120^\circ\). Since \(\angle COD = \frac{2}{3}\angle AOB\),
\[
\angle AOB + \frac{2}{3}\angle AOB = 120; \quad \frac{5}{3}\angle AOB = 120; \quad \angle AOB = \frac{3}{5}(120) = \frac{360}{5} = 72
\]
Then \(\angle COD = \frac{2}{3}(72^\circ) = 48^\circ\).

4) Every 12 months we count back we arrive at November again. Then 12, 24, 36, 48, \ldots, 96 months ago, it was November. Counting back four more months gives October, September, August, and July. Then 100 months ago it was July. The woman’s son was born in July.

5) Let \(t\) be the number of three-leaf clovers and \(f\) the number of four-leaf clovers Shannon picked. Since the total number of clovers is 17, we have \(t + f = 17\). Since the total number of leaves is 63, we have \(3t + 4f = 63\). Multiplying the first equation by 3 leads to the system
\[
\begin{cases}
3t + 3f = 51 \\
3t + 4f = 63
\end{cases}
\]
Subtracting the first equation from the second gives \(f = 63 - 51 = 12\). Then Shannon picked 12 four-leaf clovers.
6) Let $T$ be the point where $BC$ is tangent to the smaller circle. Then $\angle OTC = 90^\circ$. Also, since $AC$ is a diameter we have $\angle ABC = 90^\circ$ as well. Since $\triangle OTC$ and $\triangle ABC$ share an angle at $C$, it follows that $\angle TOC = \angle BAC$. Then $\triangle OTC$ and $\triangle ABC$ are similar. Since corresponding sides of similar triangles are proportional, $OT/OC = AB/AC$. But $AC = 2OC$. Then

$$\frac{OT}{OC} = \frac{AB}{2OC}; \quad 2(OC)(OT) = (OC)(AB); \quad 2OT = AB = 12; \quad OT = \frac{12}{2} = 6$$

Thus the smaller circle has radius 6 and it follows that the larger circle has radius 18.

7) On Thursday, Friday, Saturday, and Sunday, the girl tells only the truth. Then she cannot say “I am lying today and will tell the truth tomorrow” on any of these days. On Monday, she can say “I am lying today and will tell the truth tomorrow” because this statement would be false. (She would tell only lies on the next day.) On Tuesday, she can also say this. However, on Wednesday she cannot say this. On that day, the statement would be true but she can only tell lies on Wednesdays. Therefore, there are 2 days of the week (Monday and Tuesday) on which the girl can make that statement.

Part B

8) Since $N$ has remainder 4 when it is divided by 7, $N = 7p + 4$ for some natural number $p$. Since $N$ also has remainder 5 when divided by 12, $N = 12q + 5$ for some natural number $q$. If we subtract 4 from both sides of both equations, we obtain $N - 4 = 7p$ and $N - 4 = 12q + 1$. Thus $N - 4$ is a 7-multiple and $N - 4$ has remainder 1 when divided by 12. The 7-multiples are: 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, ... The first number on this list which has remainder 1 when divided by 12 is 49. Then $N - 4 = 49$ and hence $N = 53$. 

9) Let \( R \) be the point where \( PQ \) is tangent to the circle. Then \( OR \perp PQ \) and hence \( OR \perp PR \). Since \( OR \) is a radial segment, \( OR = 1 \). To find \( OA \), note that \( OS \perp AD \) and hence \( OS \perp AS \). Then \( OS^2 + AS^2 = OA^2 \) by the Pythagorean theorem. Since \( OS = 1 \) and \( AS = 1 \), it follows that \( OA = \sqrt{2} \). Then \( AR = \sqrt{2} - 1 \). Since \( \angle OAP = 45^\circ \) and \( \angle ARP = 90^\circ \), we have \( \angle APR = 45^\circ \). Then \( \triangle ARP \) is an isosceles triangle and \( AR = RP \). By the Pythagorean theorem,

\[
OR^2 + RP^2 = OP^2; \quad 1^2 + (\sqrt{2} - 1)^2 = OP^2;
1 + (2 - 2\sqrt{2} + 1) = OP^2; \quad 4 - 2\sqrt{2} = OP^2; \quad OP = \sqrt{4 - 2\sqrt{2}}
\]

10) In order for \( N^2 - 72 \) to be a perfect square, there must be some natural number \( M \) such that \( N^2 - 72 = M^2 \). In this case, \( N^2 - M^2 = 72 \) and in particular \( N > M \). Also, \( (N - M)(N + M) = 72 \). Then \( N - M \) and \( N + M \) are natural numbers which form a factorization of 72. Thus one of the following pairs of equations must hold:

\[
\begin{align*}
N - M &= 1, \quad N + M = 72; & N - M &= 4, \quad N + M = 18; \\
N - M &= 2, \quad N + M = 36; & N - M &= 6, \quad N + M = 12; \\
N - M &= 3, \quad N + M = 24; & N - M &= 8, \quad N + M = 9;
\end{align*}
\]

The respective solutions are

\[
\begin{align*}
N &= \frac{73}{2}, \quad M = \frac{71}{2}; & N &= 11, \quad M = 7; \\
N &= 19, \quad M = 17; & N &= 9, \quad M = 3; \\
N &= \frac{27}{2}, \quad M = \frac{21}{2}; & N &= \frac{17}{2}, \quad M = \frac{1}{2}.
\end{align*}
\]

The cases with \( N = 72/2, N = 27/2, \) and \( N = 7/2 \) are not possible because \( N \) must be a natural number. Then there are three numbers \( N \) for which \( N^2 - 72 \) is a perfect square. These are \( N = 19, N = 11, \) and \( N = 9 \).