Solutions

Problem 1 Let $SL_2(\mathbb{Z})$ be the set of all $2 \times 2$ matrices with integer entries and determinant equal to one. Find an example of matrices $A, B, C \in SL_2(\mathbb{Z})$ such that


For instance, take the matrices

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Problem 2 $A$ and $B$ are two different numbers belonging to the first forty natural numbers (1 and 40 included). Find the maximum value for the expression

$$AB - A - B.$$ 

The expression $AB/(A - B)$ is maximized when the product $AB$ is maximized and the difference $A - B$ is minimized. $A - B$ is minimized for $A = 40$ and $B = 39$. Thus, we have

$$\frac{AB}{A - B} = \frac{40 \cdot 39}{40 - 39} = 1560.$$ 

Problem 3 Let $P$ be a real polynomial of degree five. Assume that the graph of $P$ has three inflection points lying on a straight line. Calculate the ratios of the areas of the bounded regions between this line and the graph of the polynomial.

Let $A$, $B$ and $C$ be the inflection points. Let $\ell$ be the equation of the line $y = kx + n$ passing through them. If $B$ has coordinates $(x_0, y_0)$ the transformation $x' = x - x_0, \quad y' = kx - y + n$ brings $\ell$ into the $x$-axis and the point $B$ into the origin. Without loss of generality it is sufficient to consider a fifth-degree polynomial $P(x)$ with points of inflection $(b, 0), (0, 0)$, and $(a, 0)$ with $b < 0 < a$. Moreover, the third derivative of the polynomial $P(x)$ is

$$P'''(x) = kx(x - a)(x - b)$$

and

$$P(x) = \frac{k}{20} x^5 - \frac{k(a + b)}{12} x^4 + \frac{ka^3}{6} x^3 + cx + d.$$ 

By substituting the coordinates of the inflection points we find

$$d = 0, \quad a + b = 0, \quad c = \frac{7ka^4}{60}$$

and

$$P(x) = \frac{k}{20} x^5 - \frac{ka^2}{6} x^3 + \frac{7ka^4}{60} x = \frac{k}{60} x(x^2 - a^2)(3x^2 - 7a^2).$$

Since $P(x)$ is an odd function, the figures bounded by its graph and the $x$-axis are pairwise equiareal. Two of the figures with unequal areas are

$$\Omega_1 := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, \quad 0 \leq y \leq P(x)\},$$
\[ \Omega_2 := \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq a\sqrt{7/3}, \ P(x) \leq y \leq 0\}. \]

In particular, we find

\[ S_1 = S(\Omega_1) = \int_0^a P(x) \, dx = \frac{ka}{40}, \quad S_2 = S(\Omega_2) = -\int_a^{a\sqrt{7/3}} P(x) \, dx = \frac{4ka}{405} \]

and we conclude that

\[ \frac{S_2}{S_1} = \frac{32}{81}. \]

**Problem 4** Find all integer solutions of the equation

\[ x^4 + x^3 + x^2 + x = y^2 + y \]

The above equation is a quadratic equation in the variable \( y \) and we can rewrite the given equation as

\[ y^2 + y - (x^4 + x^3 + x^2 + x) = 0. \]

The roots of the above equation are

\[ y_1 = -\frac{-1 + \sqrt{1 + 4(x^4 + x^3 + x^2 + x)}}{2}, \quad y_2 = -\frac{-1 - \sqrt{1 + 4(x^4 + x^3 + x^2 + x)}}{2}. \]

The requirement that \( y \) is an integer implies that the discriminant \( \Delta = 1 + 4(x^4 + x^3 + x^2 + x) \) is a perfect square. Let us rewrite \( \Delta \) as follows

\[ \Delta = (2x^2 + x + 1)^2 - (x^2 - 2x). \]

Notice that if \( x = 0 \) or \( x = 2 \) we have \( x^2 - 2x = 0 \) and \( \Delta \) would be a perfect square. Let us consider the following cases

- Case \( x = 0 \): we have \( \Delta = 1 \) and \( y_1 = 0, \ y_2 = -1 \).
- Case \( x = 2 \): we have \( \Delta = 121 \) and \( y_1 = 5, \ y_2 = -6 \).
- Case \( x = 1 \): we have \( \Delta = 17 \) and \( \Delta \) can not be a perfect square.

For \( x \) integer and different from 0, 1, 2 we have \( x^2 - 2x > 0 \) and \( \Delta < (2x^2 + x + 1)^2 \). Since \( 2x^2 + x + 1 \geq 1 \) for every \( x \) and since \( \Delta \) has to be a perfect square, it must be

\[ \Delta \leq (2x^2 + x)^2. \]

Let us find those values of \( x \) such that the inequality

\[ (2x^2 + x + 1)^2 - (x^2 - 2x) \leq (2x^2 + x)^2 \]

is true. After rewriting the above inequality in the form \((3x + 1)(x + 1) \leq 0\) we find that

\[ -1 \leq x \leq -\frac{1}{3}. \]

The only one value for \( x \) satisfying \( \Delta \leq (2x^2 + x)^2 \) is \( x = -1 \). For \( x = -1 \) we have \( \Delta = 1 \) and \( y = 0 \) or \( y = -1 \). We conclude that the only integer solutions \((x, y)\) of the original equation are

\[ (-1, 0), \quad (0, -1), \quad (0, 0), \quad (2, -6), \quad (2, 5). \]
Problem 5 If \( n > 2 \) is an integer, show that \((n!)^2 > n^n\).

Let us prove the above statement for \( n \geq 3 \). First of all, notice that

\[
(n!)^2 = n! \cdot n! = 1 \cdot 2 \cdot (n-1) \cdot n = (1 \cdot n) \cdot (2 \cdot (n-1)) \cdot (3 \cdot (n-2)) \cdots ((n-1) \cdot 2) \cdot (n \cdot 1) = \prod_{k=1}^{n} k \cdot (n-k+1).
\]

Moreover,

\[
n^n = \prod_{k=1}^{n} k.
\]

To prove the original statement we proceed by induction. Notice that it suffices to show that

\[
k(n-k+1) > n \quad \text{for} \quad k = 2, 3, \ldots, n-1,
\]

since

\[
k(n-k+1) = n \quad \text{for} \quad k = 1
\]

and

\[
k(n-k+1) = n \quad \text{for} \quad k = n.
\]

Now, for \( k = 2, 3, \ldots, n-1 \) we have

\[
k(n-k+1) > n \iff kn - k^2 + k > n
\]

\[
\iff kn - n > k^2 - k
\]

\[
\iff n(k-1) > k(k-1) \quad \text{and since} \quad k \neq 1
\]

\[
\iff n > k
\]

which is true for our problem since \( 2 \leq k \leq n-1 \). We conclude that since \( n \geq 3 \) there exists at least one \( k \) such that \( 2 \leq k \leq n-1 \) and \( k(n-k+1) > n \).

Problem 6 Let \( n \) be a positive integer and \( f : [0, 1] \rightarrow \mathbb{R} \) be a continuous function such that

\[
\int_0^1 x^k f(x) \, dx = 1
\]

for every \( k \in \{0, 1, \cdots, n-1\} \). Prove that

\[
\int_0^1 f^2(x) \, dx \geq n^2.
\]

There exists a polynomial \( p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} \) satisfying

\[
\int_0^1 x^k p(x) \, dx = 1 \quad \text{for all} \quad k = 0, 1, \cdots, n-1.
\]

It follows that for all \( k = 0, 1, \cdots, n-1 \)

\[
\int_0^1 x^k (f(x) - p(x)) \, dx = 0
\]
and hence
\[ \int_0^1 p(x)(f(x) - p(x)) \, dx = 0. \]

Then, we can write
\[
\int_0^1 (f(x) - p(x))^2 \, dx = \int_0^1 f(x)(f(x) - p(x)) \, dx \\
= \int_0^1 f^2(x) \, dx - \sum_{k=0}^{n-1} a_{k+1} \int_0^1 x^k f(x) \, dx,
\]
and since the first integral is non-negative we get
\[ \int_0^1 f^2(x) \, dx \geq a_1 + a_2 + \cdots + a_n. \]

To complete the proof we show that
\[ a_1 + a_2 + \cdots + a_n = n^2 \]
for the coefficients of the polynomial \( p \). The defining property of \( p \) can be rewritten as
\[ \frac{a_1}{k+1} + \frac{a_2}{k+2} + \cdots + \frac{a_n}{k+n} = 1, \quad 0 \leq k \leq n - 1. \]

Equivalently, the function
\[ r(x) = \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_n}{x+n} - 1 \]
has 0, 1, \ldots, \( n-1 \) zeroes. We write \( r \) in the form
\[ r(x) = \frac{q(x) - (x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)}, \]
where \( q \) is a polynomial of degree \( n-1 \). Observe that the coefficients of \( x^{n-1} \) in \( q \) is equal to \( a_1 + a_2 + \cdots + a_n \). Moreover, the numerator has 0, 1, \ldots, \( n-1 \) as zeroes and since
\[ \lim_{x \to \infty} r(x) = -1 \]
we must have
\[ q(x) = (x+1)(x+2)\cdots(x+n) - x(x-1)\cdots(x-(n-1)). \]

This expression for \( q \) shows that the coefficient of \( x^{n-1} \) in \( q \) is
\[ \frac{n(n+1)}{2} + \frac{(n-1)n}{2}. \]

It follows that \( a_1 + a_2 + \cdots + a_n = n^2. \)