Solutions to the final exam for M32Q
May 2010

Problem 1

1. An integrating factor is \( \mu(t) = e^{t^2/2} \) (1 point) and we obtain
\[
 e^{t^2/2} y(t) = \int t^3 e^{t^2/2} dt.
\]

Integrating by parts (1 point) we have
\[
\int t^3 e^{t^2/2} dt = \int t^2 \frac{d}{dt} e^{t^2/2} dt = t^2 e^{t^2/2} - 2 \int t e^{t^2/2} dt = (t^2 - 2)e^{t^2/2} + C.
\]

The above integral can be also computed by means of the substitution
\( x = t^2/2 \) and then by integration by parts. The general solution is then given by
\[
y(t) = t^2 - 2 + Ce^{-t^2/2}.
\]

Substituting \( t = 0 \) in the above equation we find (1 point)
\[
C = 2
\]
and the solution of the initial problem is (1 point)
\[
y(t) = t^2 - 2 + 2(e^{-t^2/2} - 1).
\]

2. By separation of variables we obtain
\[
e^{-y} dy = \frac{dt}{t^2}
\]
and integrating we have (1 point)
\[
-e^{-y} = \frac{1}{t} + \tilde{C}.
\]

Solving the above equation for \( y \) and setting \( C = -\tilde{C} \) we find the general solution (1 point)
\[
y(t) = \ln \left( \frac{t}{Ct - 1} \right).
\]

The presence of a logarithm requires a short discussion on the domain of definition of the above solution. Since the argument of the logarithm must be a positive function and since in the formulation of the problem \( t > 0 \) we have to require that \( Ct - 1 > 0 \). Clearly, \( C \) cannot be negative since the argument of the logarithm would be also negative and for the same reason the case \( C = 0 \) is also ruled out. Therefore, we conclude the solution we found is defined for \( t > 1/C \).
3. By means of the ansatz $y(t) = e^{\lambda t}$ we get the characteristic polynomial

$$p(\lambda) = \lambda^2 - 2\lambda + 2$$

and the roots of the equation $p(\lambda) = 0$ are (1 point)

$$\lambda \pm = 1 \pm i.$$

Notice that

$$e^{\lambda+t} = e^t e^{it}, \quad e^{\lambda-t} = e^t e^{-it}$$

Two particular solutions of the original differential equations are

$$y_1(t) = \frac{e^{\lambda+t} + e^{\lambda-t}}{2} = e^t \cos t,$$

$$y_2(t) = \frac{e^{\lambda+t} - e^{\lambda-t}}{2i} = e^t \sin t.$$  

We conclude that the general solution is (1 point)

$$y(t) = e^t (C_1 \cos t + C_2 \sin t).$$

4. First of all, we have to find the solution of the homogeneous equation

$$\ddot{y} - 4y = 0.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 4\lambda + 4$$

and $\lambda = -2$ is a double root of the equation $p(\lambda) = 0$. According to the theory seen in class two particular solutions will be

$$y_1(t) = e^{2t}, \quad y_2(t) = te^{2t}.$$  

The solution of the homogeneous equation is (1 point)

$$y_h(t) = (C_1 + C_2 t)e^{2t}.$$  

To find a particular solution $Y(t)$ of the nonhomogeneous problem one can use variation of parameters, the method of undetermined coefficients or the general formula

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt$$

where $g(t)$ is the source of non homogeneity (in the present case $g(t) = 2e^t$) and $W(y_1, y_2)$ is the Wronskian which is computed to be $W(y_1, y_2) = e^{4t}$. Hence, we find (2 points)

$$Y(t) = -2e^{2t} \int te^{-t} dt + 2te^{2t} \int e^{-t} dt,$$

$$= 2e^{2t} \int t e^{-t} dt - 2te^{2t} e^{-t},$$

$$= 2e^{2t} \left( te^{-t} - \int e^{-t} dt \right) - 2te^t,$$

$$= 2te^t - 2e^{2t} \int e^{-t} dt - 2te^t = -2e^{2t} \int e^{-t} dt = 2e^t.$$
The general solution is
\[ y(t) = y_h(t) + Y(t) = (C_1 + C_2t)e^{2t} + 2e^t. \]

**Problem 2**

1. (3 points) Apply the chain rule by considering \( \omega \) as a function of the angle \( \vartheta \). Then, for \( \omega = \omega(\vartheta) \) we have
\[
\frac{d^2 \vartheta}{dt^2} = \frac{d}{dt} \left( \frac{d\vartheta}{dt} \right) = \frac{d\omega}{dt} \frac{d\vartheta}{dt} = \omega \frac{d\vartheta}{dt}.
\]

2. The pendulum equation becomes a first order separable nonlinear differential equation, namely (7 points)
\[
\omega \frac{d\omega}{d\vartheta} + K \sin \vartheta = 0, \quad K > 0. \tag{1}
\]

After separation of variables we have
\[
\int \omega d\omega = -K \int \sin \vartheta d\vartheta,
\]
and the implicit solution of (1) is
\[
\frac{\omega^2}{2} = K \cos \vartheta + C.
\]

**Problem 3**

The problem can be solved in two steps, namely

1. Construct the Laplace transform of the given equation
\[
\mathcal{L}[y(t)] = Y(s) \quad \mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 2 \quad (4 \text{ points})
\]

Notice that the discontinuous function \( g(t) \) can be written more compactly as (2 points)
\[
g(t) = [1 - H_1(t)](1 - t) = 1 - t + H_1(t)(t - 1)
\]
where \( H_1(t) \) is the Heaviside function. The Laplace transform of \( g(t) \) is (2 points)
\[
\mathcal{L}[g(t)] = \mathcal{L}[1] - \mathcal{L}[t] + \mathcal{L}[H_1(t)(t - 1)] = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2} = \frac{1}{s} - \frac{1 - e^{-s}}{s^2}
\]
where we used the results
\[
\mathcal{L}[x^n] = \frac{n!}{s^{n+1}} \quad n \in \mathbb{N}, \quad \mathcal{L}[H_1(t)f(t - c)] = e^{-cs} F(s)
\]
with $F(s)$ the Laplace transform of $f(x)$. Finally, the Laplace transform of our original differential equation is (2 points)

$$s^2Y(s) - s - 2 - Y(s) = \frac{1}{s} - \frac{1 - e^{-s}}{s^2}.$$ 

Solving this algebraic equation for $Y(s)$ we get the Laplace transform of the solution of our initial value problem, namely

$$Y(s) = \frac{s + 2}{s^2 - 1} + \frac{1}{s(s^2 - 1)} - \frac{1 - e^{-s}}{s^2(s^2 - 1)} = \frac{s^3 + 2s^2 + s - 1 + e^{-s}}{s^2(s^2 - 1)}.$$ 

2. The last step is to find the inverse Laplace transform of $Y(s)$. To this purpose it is convenient to express $Y(s)$ in terms of sums of partial fractions. In particular, we have

$$\begin{align*}
\frac{s + 2}{s^2 - 1} &= 3 \frac{1}{2s - 1} - 1 \frac{1}{2s + 1} \quad (2 \text{ points}) \\
\frac{1}{s(s^2 - 1)} &= 1 \frac{1}{2s - 1} + 1 \frac{1}{2s + 1} - \frac{1}{s} \quad (2 \text{ points}) \\
\frac{1}{s^2(s^2 - 1)} &= 1 \frac{1}{2s - 1} - \frac{1}{2s + 1} - \frac{1}{s^2} \quad (2 \text{ points}).
\end{align*}$$

By rewriting $Y(s)$ as follows (4 points)

$$Y(s) = \frac{s + 2}{s^2 - 1} + \frac{1}{s(s^2 - 1)} - \frac{1}{s^2(s^2 - 1)} + \frac{e^{-s}}{s^2}$$

$$= \frac{1}{s^2} - \frac{1}{s} + \frac{3}{2s - 1} + \frac{1}{2s + 1} + \frac{1}{2s - 1} - \frac{1}{2s + 1} - \frac{e^{-s}}{s^2}$$

By using the linearity of the inverse Laplace transform we have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{s - 1}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{e^{-s}}{s - 1}\right] - \frac{1}{2} \mathcal{L}^{-1}\left[\frac{e^{-s}}{s + 1}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right].$$

Taking into account that

$$\begin{align*}
\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] &= t, \quad \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \quad \mathcal{L}^{-1}\left[\frac{1}{s - 1}\right] = e^t, \quad \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] = e^{-t}, \\
\mathcal{L}^{-1}\left[\frac{e^{-s}}{s - 1}\right] &= H_1(t)e^{-t}, \quad \mathcal{L}^{-1}\left[\frac{e^{-s}}{s + 1}\right] = H_1(t)e^{-t-1}, \quad \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = H_1(t)(t-1)
\end{align*}$$

and the solution is given by

$$y(t) = t - 1 + \frac{3}{2}e^t + \frac{1}{2}e^{-t} + H_1(t)\left(\frac{1}{2}e^{t-1} - \frac{1}{2}e^{-t-1} - t + 1\right).$$

**Problem 4**

The problem consists of three parts
1. In order to derive the first order linear system associated to our third order differential equation we set (5 points)

\[ x_1(t) = y(t), \quad x_2(t) = \frac{dy}{dt}, \quad x_3(t) = \frac{d^2y}{dt^2} \]

and we get

\[ x_1' = x_2, \quad x_2' = x_3, \quad x_3' = -2x_3 + 4x_2 + 8x_1 \]

where \( \cdot \) denotes the first order derivative with respect to \( t \). Hence, the first order system is

\[ x' = Ax \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 4 & -2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \]

2. (10 points). The eigenvalues of \( A \) are found by solving the characteristic equation

\[ 0 = \det(A - \lambda I_3) = -\lambda^2(2 + \lambda) + 4\lambda + 8 = (2 - \lambda)(\lambda + 2)^2 \]

whose roots are

\[ \lambda_1 = \lambda_2 = -2, \quad \lambda_3 = 2. \]

Notice that the eigenvalue \(-2\) has algebraic multiplicity 2 whereas the eigenvalue \(+2\) has algebraic multiplicity 1. We compute the geometric multiplicities. To this purpose we have to determine the dimension of the associated eigenspace. Starting with \( \lambda_3 \) we have

\[ Av = 2v \implies \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \]

Row reduction gives

\[ -2v_1 + v_2 = 0, \quad 2v_2 - v_3 = 0 \]

that is

\[ v_2 = 2v_1, \quad v_3 = 2v_2 = 4v_1. \]

Therefore,

\[ v = v_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \text{ and w.l.o.g. we choose } v_1 = 1 \implies v_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}. \]

Concerning \( \lambda_1 = \lambda_2 \) we have

\[ Au = -2u \implies \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \]

Row reduction gives

\[ 2u_1 + u_2 = 0, \quad 2u_2 + u_3 = 0 \]
that is
\[ u_2 = -2u_1, \quad u_3 = -2u_2 = 4u_1. \]

Therefore,
\[
\mathbf{u} = u_1 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \text{ and w.l.o.g. we choose } u_1 = 1 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.
\]

Since the eigenspace associated to the eigenvalue \(-2\) has geometric multiplicity less than its algebraic multiplicity we conclude that the matrix \(A\) cannot be diagonalized. However, we can bring \(A\) into its Jordan form by finding a generalized eigenvector \(w\) for the matrix \(A\). To this purpose we consider the equation
\[
(A + 2I)w = u,
\]
that is
\[
\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.
\]

Taking the corresponding augmented matrix and by means of elementary row operations we obtain
\[ 2w_1 + w_2 = 1, \quad 2w_2 + w_3 = -2 \]
that is
\[ w_2 = 1 - 2w_1, \quad w_3 = -2 - 2w_2 = -4 + 4w_1. \]

Therefore,
\[
w = \begin{pmatrix} w_1 \\ 1 - 2w_1 \\ -4 + 4w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.
\]

Without loss of generality we can choose \(w_1 = 0\) and the generalized eigenvector is
\[
\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}.
\]

By means of the vectors \(\mathbf{v}_1, \mathbf{v}_2\) and \(\mathbf{v}_3\) we can construct a matrix
\[
T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ -4 & 4 & 4 \end{pmatrix} \text{ with } T^{-1} = \begin{pmatrix} 1/4 & 0 & -1/4 \\ 3/4 & -1/4 & -1/16 \\ 1/4 & 1/4 & 1/16 \end{pmatrix}
\]
such that the Jordan form of \(A\) is
\[
J = T^{-1}AT = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

If we introduce the transformation \(\mathbf{x} = Ty\) the original system \(\mathbf{x}' = Ax\) becomes
\[
\mathbf{y}' = J\mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.
\]
We have to solve the first order differential equations

\[ \frac{dy_1}{dt} = -2y_1, \quad \frac{dy_2}{dt} = y_1 - 2y_2, \quad \frac{dy_3}{dt} = 2y_3. \]

The general solutions are

\[ y_1(t) = C_1 e^{-2t}, \quad y_2(t) = (C_1 t + C_2)e^{-2t}, \quad y_3(t) = C_3 e^{2t}. \]

Finally, the solution of the first order linear system \( x' = Ax \) is

\[ x = Ty, \]

\[ = C_1 \left[ \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right] e^{-2t} + C_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}. \]

3. (5 points) The general solution of the third order linear differential equation with constant coefficients will be

\[ y(t) = x_1(t) = C_1 t e^{-2t} + C_2 e^{-2t} + C_3 e^{2t}. \]