Problem 1: (13 points)

Let $X$ be a metric space with metric $d$.

1. (2 points) Suppose that $a, b,$ and $c$ are positive real numbers and that $a \leq b + c$. Show that
   \[ \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c} \]

2. (3 points) Prove that
   \[ \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \]
   defines a metric on $X$. Moreover, show that the metric $\tilde{d}$ is bounded by one, i.e. $\tilde{d}(x, y) < 1$ for all $x, y \in X$.

3. (6 points) Prove that every open set in the metric space $(X, \tilde{d})$ is open in the metric space $(X, d)$ and vice versa.

4. (2 points) In an arbitrary metric space $(X, \hat{d})$, is it true or false that if $x \neq y$ with $x, y \in X$ and if $\epsilon \neq \delta$ with $\epsilon, \delta > 0$, then $B_\epsilon(x) \neq B_\delta(x)$?

Problem 2: (6 points)

1. (2 points) Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}^n$, i.e. the Euclidean space of dimension $n$, such that $(x_k)_{k \in \mathbb{N}} \to x$ as $k \to \infty$. Show that $(\|x_k\|)_{k \in \mathbb{N}} \to \|x\|$ as $k \to \infty$.

2. (2 points) Let $(X, \|\cdot\|_X)$ be a Banach space and $u_1, u_2, \cdots$ elements of $X$. Let $\sum_{n=1}^\infty u_n$ be absolutely convergent. Show that, $\sum_{n=1}^\infty u_n$ is convergent and
   \[ \|\sum_{n=1}^\infty u_n\| \leq \sum_{n=1}^\infty \|u_n\|. \]

3. (2 points) Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Moreover, let $T_1, T_2, \cdots$ be bounded linear operators from $X$ to $Y$. Suppose that $\sum_{n=1}^\infty \|T_n\|$ is convergent. Show that, $\sum_{n=1}^\infty T_n$ is convergent and
   \[ \|\sum_{n=1}^\infty T_n\| \leq \sum_{n=1}^\infty \|T_n\|. \]

Problem 3: (7 points)

Let $T$ be an operator in $B(X, Y)$ with $Y$ Banach space. Suppose that $\|T\| < 1$. Then, the operator $I_X - T$ has a bounded inverse $(I_X - T)^{-1}$ where $I_X$ denotes the identity operator on $X$.

Problem 4: (12 points)

Consider the space $C([0, 1])$ and the metric
   \[ d(f, g) = \int_0^1 |f(x) - g(x)| \, dx \quad \forall f, g \in C([0, 1]). \]

Show that $C([0, 1])$ equipped with the above metric is incomplete.

GOOD LUCK!